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Oscillations of Viscoelastic Mechanical Systems with Finite Freedom

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Abstract

The study investigates the natural and externally excited oscillations of viscoelastic mechanical systems possessing a finite number of degrees of freedom. Based on Lagrange's second-order equations, the dynamical model of systems with dissipation was derived. Particular attention is paid to periodic as well as transient forced vibrations in multi-degree-of-freedom structures. The system of equations of motion is written in matrix form relative to the matrix-column. $\{X\} = \text{colon}(x_1, \ldots, x_n)$. The characteristic parameters $\lambda_k = \omega_{Rk} - i\omega_{Ik}$, $\lambda_{n-k} = \omega_{Rk} + i\omega_{Ik}$, $(k = 1, \ldots, n)$ were found, where $\omega_{Ik} > 0$, and $\omega_{Rk} > 0$, are real numbers called damping coefficients. The attenuation decrement ratio was also determined. The Fourier transform method is used to analyze non-stationary oscillations in mechanical systems.

Keywords: Mechanical systems, Non-stationary oscillations, Movements, Decrement of damping, Finite number of degrees of freedom.

1 | Introduction

In modern mechanical engineering, machines and technological units show a continuous tendency toward increased power capacity and higher operational parameters [1–3]. This trend also characterizes equipment used across various industrial sectors, including forestry and processing. As operational intensity increases, machinery vibration increases, leading to higher dynamic loads on structural components [4], [5].

This intensification accelerates wear processes, increases the probability of failures, and negatively influences the performance quality of machines and mechanisms. To study these effects, consider a linear mechanical system with nnn degrees of freedom oscillating in the vicinity of a stable equilibrium configuration [6], [7]. Let us imagine that a linear mechanical system with n degrees of freedom performs small oscillations around

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a stable equilibrium point [8], [9]. We will describe the system's dynamic state in terms of generalized coordinates; the coordinates will be zero in the equilibrium state.

The dynamic state of mechanical systems with concentrated parameters, their stability properties, and the factors influencing vibrational processes were analyzed from a scientific perspective.

2 | Problem Statement and Methodology for Solving It

The work focuses on mechanical systems with lumped parameters. Using Lagrange's formalism, the governing equations of motion are developed.

$$\frac{d}{dt}\left(\frac{dT}{dx_j}\right)+\frac{dD}{dx_j}+\frac{dV}{dx_j}=f_j(t), \qquad j=1,2,3\ldots,n. \tag{1}$$

The generalized external force $f_j(t)$ acting at each coordinate may either arise from imposed motion or be an externally applied dynamic load. The system of equations is expressed in compact matrix form:

$$[M]\{\ddot{X}\}+[C]\{\dot{X}\}+[K]\{X\}=\{f\}, \tag{2}$$

where $[M],[C]$ and $[K]$ denote the inertia, damping, and stiffness matrices of order n.. The perturbation is described by the matrix of the column $\{f\}$. The physical meaning of the matrix coefficients is as follows: M_{jk} – is the component of the amount of motion along j at unit velocity along k, C_{jk} – the damping force along j at unit velocity along k, K_{jk} – the elastic force along j due to unit displacement along k.

For oscillations defined by acceleration, this expression looks like this:

$$a=\sum_{\kappa}A_{\kappa}\cos(\omega_{\kappa}t-\Phi_{\kappa})=R_e\sum_{\kappa}A_{\kappa}e^{i(\omega_{\kappa}t-\Phi_{\kappa})}=\sum_{\kappa}A_{\kappa}\sin(\omega_{\kappa}t-\theta_{\kappa})=R_e\sum_{\kappa}A_{\kappa}e^{i(\omega_{\kappa}t-\theta_{\kappa})}, \tag{3}$$

where $\omega_{\kappa}=2\pi f_{\kappa}$, ω_{κ} – is the angular frequency, f_{κ} – and is the corresponding frequency. If all f_{κ} excitation frequencies are integer multiples of a base frequency, the oscillations are strictly periodic; otherwise, the motion becomes quasi-periodic.

Accelerations can be determined from velocities by multiplying them by $2\pi f$, or in complex form, by $i2\pi f$ where the imaginary unit iii introduces a phase shift of 90^0.

Steady-state oscillations are considered an approximate mathematical model. If the amplitudes A_{κ} and phase parameters Φ_{κ} Θ_{κ} vary slowly, the resulting motion may be regarded as quasi-periodic. For strictly periodic oscillations, typically only amplitude values or their RMS equivalents are taken into account, while phase angles are often omitted.

Instantaneous accelerations and the magnitudes of harmonics at the same frequency are summed vectorially. The phase difference plays a key role in resonance analysis, particularly in the experimental determination of resonant frequencies. In the vicinity of resonance, the phase changes more rapidly than the frequency itself. The vector of external forces $\{f\}=0$, *, Eq. (2)*, describes the system's free vibrations when $\{f\}\neq 0$, it corresponds to forced oscillations.

Free oscillations of dissipative systems. Let us consider a linear dissipative system whose motion is described by the matrices [M], [C], and [K]. The solution of *Eq. (3)* can be represented as

$$\{X(t)\}=\{W\}e^{\lambda t}, \tag{4}$$

where λ is a complex number, W is a complex matrix of numbers, column. The numbers λ are called characteristic indicators, and the numbers $i\lambda$ (or $-i\lambda$) are called complex frequencies. Characteristic indicators must be the roots of the characteristic equation.

$$\det[M]\lambda^2+[C]\lambda+[K]\big)=0, \quad \textbf{(5)}$$

The defining expression in its expanded form can be presented as follows:

$$\begin{vmatrix} a_{11}\lambda^2+b_{11}\lambda+c_{11} & a_{12}\lambda^2+b_{12}\lambda+c_{12} & \cdots & a_{1n}\lambda^2+b_{1n}\lambda+c_{1n} \\ a_{21}\lambda^2+b_{21}\lambda+c_{21} & a_{22}\lambda^2+b_{22}\lambda+c_{22} & \cdots & a_{2n}\lambda^2+b_{2n}\lambda+c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}\lambda^2+b_{n1}\lambda+c_{n1} & a_{n2}\lambda^2+b_{n2}\lambda+c_{n2} & \cdots & a_{nn}\lambda^2+b_{nn}\lambda+c_{nn} \end{vmatrix}=0.$$

A total of 2n characteristic indicators characterizes a mechanical system with n degrees of freedom. $\lambda_1,\lambda_2,...,\lambda_{2n}$.

If these characteristic indicators form the simple roots of *Eq. (5)*, then the general solution of *Eq. (2)* is represented as a superposition of the particular solutions given in *Eq. (5)*.

$$\{X(t)\}=\sum_{k=1}^{2n}C_k\{W_k\}e^{-i\lambda_k t}. \quad \textbf{(6)}$$

Here $C_к$ are arbitrary complex constants, $W_к$ which are columns of the numerical matrix.

Let's express the characteristic levels as follows:

$$\lambda_k=\omega_{Rk}-i\omega_{Ik},\quad \lambda_{n-k}=\omega_{Rk}+i\omega_{Ik},\quad (k=1,\ldots,n). \quad \textbf{(7)}$$

Here $\omega_{Iк}$, $\omega_{Rк}$ represent the damping coefficients and the natural frequencies of the dissipative system, respectively. If $\{W_к\}$ and $\lambda_к$ satisfy *Eq. (5)*, then their complex conjugate values are also solutions for this equation. In the absence of damping, all roots are located on the imaginary axis, and in the presence of damping, their real parts shift slightly to the negative side, i.e., to the left half-plane.

The corresponding $2n$ eigenvectors fulfill the orthogonality relations:

$$\begin{aligned}&(\omega_R+\omega_I)X_R^T[M]X_I+X_R^T[C]X_I=0.\\&X_R^T[M]X_I+\omega_R\omega_I X_R^T[M]X_I=0,\end{aligned} \quad \textbf{(8)}$$

where the superscript T denotes transposition.

When $\omega_R\neq\omega_I$, orthogonality can also be ensured for multiple roots by appropriately choosing eigenvectors.

The damping coefficient has a unit of measurement $\omega_I C^{-1}$. To represent the degree of damping, you can introduce a convenient dimensionless parameter, such as:

$$\delta_{\Omega к}=\frac{\omega_{Iк}}{\Omega},\ \delta_{\omega Rк}=\frac{\omega_{Iк}}{\omega_{Rк}}. \quad \textbf{(8)}$$

With the help of the logarithmic decrement, it is possible to draw accurate conclusions about the degree of attenuation, dynamic stability, and changes in amplitude around resonance. $\delta_{\pi к}=2\pi\delta_{\omega Rк}$.

The attenuation decrement is one of the main parameters characterizing the intensity of oscillations around dynamic stability and resonance states.

$$\delta_{eк}=e^{\delta_{\pi к}}\frac{x_j(\pi\omega_{Rк}m)}{x_j\big(2\pi\omega_{Rк}(m+1)\big)},\ (m=1,2,\ldots).$$

If $\omega_{I\kappa}$ it is positive, then $\delta_{\pi\kappa}$ it represents the degree of decrease in the amplitude of oscillations, respectively. The following expression relates the attenuation coefficient to the amplitude ratio.

$$\omega_I = \frac{\omega_R}{2\pi}\ln\frac{A_{n+1}}{A_n}.$$

Problem of finding complex eigenvalues.

For systems with symmetric mass, damping, and stiffness matrices, $[M],[C],[K]$ the task reduces to finding the complex eigenvalues.

$$\lambda_\kappa = \omega_{R\kappa} - i\omega_{I\kappa}.$$

and the corresponding complex eigenvectors

$$X_\kappa = X_{R\kappa} + iX_{I\kappa}.$$

which satisfy *Eq. (5)*. Compared with the undamped case, determining complex eigenvalues is significantly more complicated and has been studied less extensively. In a conservative system (*Fig. 1-a*), all characteristic indicators are purely imaginary and differ from the system's natural frequencies only by a factor of $+i$.

All particular solutions are periodic functions of time, and the motion in the general case is stationary (almost periodic). If the system is dissipative and completely dissipative, then all characteristic indicators lie in the lower half-plane of the complex plane (*Fig. 1-b*).

When dissipation is incomplete, some eigenvalues remain on the imaginary axis while others shift to the left, so the system includes periodic modes corresponding to undamped degrees of freedom. If dissipation is negative, certain eigenvalues acquire positive real parts (*Fig. 1-d*), causing the corresponding partial and general solutions to grow unboundedly over time.

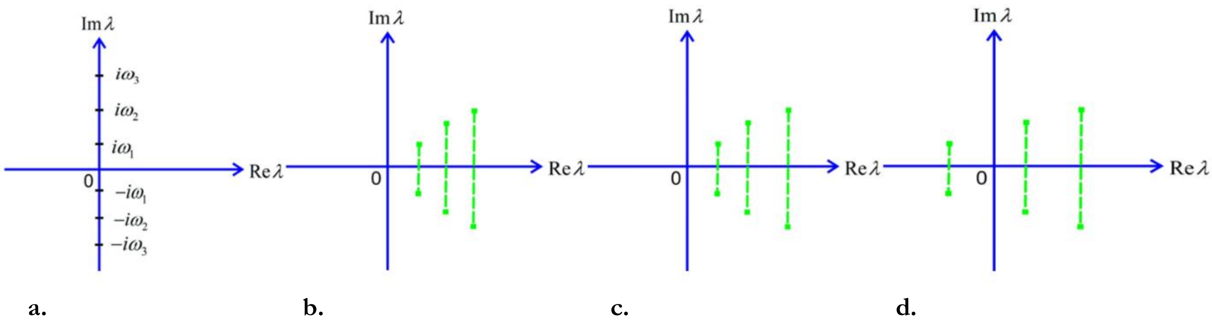

Fig. 1. Position of characteristic indicators for conservative, dissipative, and negatively damped mechanical systems; a. conservative, b. with complete dissipation, c. with incomplete dissipation, and d. with negative dissipation.

In a stationary problem, the vector function $\{f\}$ changes according to the harmonic law $\{f\}=\{F\}e^{-i\omega t}$ with a given frequency and amplitude $\{F(\omega)\}$.. Initial conditions are not set. Instead, the solution must satisfy the periodicity condition at the excitation frequency $\omega:\{x(t)\}=\{w\}e^{i\omega t}$. . This requirement leads to an algebraic system of equations formulated in terms of the complex components of the unknown vector. $\{W\}$:

$$(-\omega^2[M]+i\omega[C]+[K])\{W\}=\{F\}. \quad (9)$$

The system *Eq. (9)* with complex coefficients can be solved, for example, by the Gauss method. A linear oscillatory system is considered, the position of which in space is determined by generalized coordinates

$q_i(i=1,2,...,n).$ It is assumed that two types of generalized forces act on the system. Firstly, these are forces linearly dependent on the generalized coordinates, and this dependence is hereditary:

$$\bar{Q}_i(t)=\sum_{j=1}^{n}\left[C_{ij}q_j(t)-\int_{-\infty}^{t}R_{ij}(t-s)q_j(s)ds\right]. \tag{10}$$

Here, $\bar{Q}_i$ – generalized forces of the hereditary type C_{ij} – are such known constants that the quadratic form $\sum_{j=1}^{n}C_{ij}q_iq_j$ is positively determined, and R_{ij} – are known influence functions. Secondly, these are generalized forces, obviously dependent on time, and this dependence is harmonic. Due to the linearity of the problem, it is sufficient to consider the case when all apparently time-dependent generalized forces have equal periods and phases:

$$Q_i(t)=Q_{i0}\sin(pt),(i=1,\dots,n). \tag{11}$$

The Lagrange *Eq. (11)* for the system under consideration has the form.

$$\sum_{j=1}^{n}\left[a_{ij}\ddot{q}_j(t)+c_{ij}q_j(t)-\int_{-\infty}^{t}R_{ij}(t-s)q_j(s)ds\right]=Q_{i0}\sin(pt). \tag{12}$$

In them - coefficients of a positively defined quadratic form $\frac{1}{2}\sum_{i,j=1}^{n}a_{ij}q_iq_j$. The problem consists of finding the periodic solution of the system *Eq. (12)*. We reduce the system *Eq. (12)* to the normal coordinates of the elastic system obtained from *Eq. (12)* at $R_{ij}=0.$ For this purpose, we introduce the transformation of generalized coordinates. $q_j=\sum_{k=1}^{n}b_{ik}\Theta_k$, where Θ_k – are the normal coordinates, b_{jk} – are the transformation matrix coefficients, and $\det[b_{jk}]\neq 0.$ the system *Eq. (12)* in the coordinates Θ_k takes the form

$$\ddot{\Theta}_i+\omega_i^2\mathbf{J}_i-\int_{-\infty}^{t}\sum_{k=1}^{n}\bar{R}_{ik}(t-s)\Theta_k(s)ds=Q_{i0}\sin(pt), \qquad (i=1,\dots,n), \tag{13}$$

where ω_i – are the natural frequencies of the elastic system, and Θ_{0i} – are the amplitudes of the generalized forces corresponding to the normal coordinates, each of which is determined by the relation $\Theta_{0i}=\sum_{k=1}^{n}b_{ki}\Theta_{0k}$, where $\bar{R}_{ik}$ – is the influence function: $\bar{R}_{ik}=\sum_{j=1}^{n}R_{ij}b_{jk}$. The periodic solution of the system *Eq. (13)* is sought in the form

$$\Theta_i(t)=A_{0i}\sin(pt+\varphi_i), \tag{14}$$

where A_{0i},φ_i – the constants are determined from the system of transcendental equations:

$$\begin{aligned}(\omega_i^2-p^2)A_{0i}\cos\varphi_i-\sum_{k=1}^{n}(U_{cik}\cos\varphi_k+U_{sik}\sin\varphi_k)A_{0k}=\Theta_{0i},\\(\omega_i^2-p^2)A_{0i}\sin\varphi_i-\sum_{k=1}^{n}(U_{cik}\sin\varphi_k+U_{sik}\cos\varphi_k)A_{0k}=\Theta_{0i}.\end{aligned} \tag{15}$$

there

$$U_{cik}=\int_0^{\infty}\bar{R}_{ik}(s)\cos(ps)ds, \qquad U_{sik}=\int_0^{\infty}\bar{R}_{ik}(s)\sin(ps)ds.$$

The system *Eq. (15)* was obtained by substituting *Eqs. (14)* and *(13)* and comparing the coefficients for sinpt and cospt. In this case, identity is used.

$$\int_{-\infty}^{t} R(t-s)\sin psds = U_c(p)\sin pt - U_s(p)\cos pt,$$

where

$$U_c(p) = \int_0^{\infty} R(s)\cos psds, \qquad U_s(p) = \int_0^{\infty} R(s)\sin psds.$$

The system *Eq. (15)* can be considered as a linear algebraic system with respect to the unknowns $A_i = A_{0i}\cos\varphi_i$, $B_i = A_{0i}\sin\varphi_i$. After determining A_i, B_i , we will find the sought constants from the relations.

$$A_{0i} = \sqrt{A_i^2 + B_i^2},\ \varphi_i = \arctan\frac{B_i}{A_i}.$$

The problem is significantly simplified in the case when the influence functions R_{ij} are proportional, and the proportionality coefficients are the generalized stiffnesses C_{ij} , i.e., in the case when

$$\bar{Q}_i(t) = \sum_{j=1}^{n} C_{ij}\left[q_j(t) - \int_{-\infty}^{t} R_{ij}(t-s)q_j(s)ds\right].$$

An example of such a system is an absolutely rigid body mounted on shock absorbers with identical hereditary-elastic parameters. In the considered case, the system *Eq. (13)* takes the form.

$$\ddot{\Theta}_i + \omega_i^2\left[\Theta_i(t) - \int_{-\infty}^{t} R(t-s)\Theta_i(s)ds\right] = Q_{i0}\sin pt \qquad (i = 1,\dots,n).$$

The solution of this system is expressed by *Formula (14)*, in which

$$A_{0i} = \frac{\Theta_{0i}}{\sqrt{[p^2 - \omega_i^2(1-U_c)]^2 + \omega_i^4 U_c^2}}, \qquad \varphi_i = \arctan\frac{\omega_i^2 U_s}{p^2 - \omega_i^2(1-U_c)}.$$

Consider the forced oscillations of a system with n degrees of freedom described by *Eq. (11)*. The application of the Fourier transform to differential *Eq. (11)* leads to a system of linear algebraic equations.

$$([M]p^2 + [C]p + [K])q_*(p) = F_*(p)([M]p + [C])q_0 + [M]\dot{q}_0, \qquad \textbf{(16)}$$

where, $\{q_*(\varphi)\}$ and $\{F_*(\varphi)\}$ are matrices - image columns corresponding to the matrices $\{q(t)\}$ and $\{F(t)\}$, and the vectors $\{q_0\}$ and $\{\dot{q}_0\}$ are defined by the initial conditions. The resulting system is solved using the Gauss method, isolating the main element. Applying the inverse Fourier transform gives the desired solution. Apply to one-dimensional equations of motion written in the form.

$$M\ddot{y} + C\dot{y} + Ky = F\delta(t). \qquad \textbf{(17)}$$

Fourier transform. For this, we multiply the equation by $\exp(-ipt)$ and integrate within. $-\infty < t < \infty$.

$$M\int_{-\infty}^{\infty} e^{-ipt}\frac{d^2y}{dt^2}dt + C\int_{-\infty}^{\infty} e^{-ipt}\frac{dy}{dt}dt + K\int_{-\infty}^{\infty} e^{-ipt}ydt = \int_{-\infty}^{\infty} Fe^{-ipt}\delta(t)dt.$$

or after integration by parts, $(-Mp^2 + iCp + K)\tilde{y} = F\tilde{y}(p)$ from where

$$\tilde{y}(p) = \frac{F}{K - Mp^2 + iCp}. \quad \textbf{(18)}$$

To determine $y(t)$ the function from the found function, $\tilde{y}(p)$ one can use the subtraction method or decomposition into elementary fractions, as well as the inverse Fourier transform.

3 | Conclusions

Thus, the work examines the dynamic state of mechanical systems with concentrated parameters. Using Lagrange's equations, we obtain the equations of motion. If the system is dissipative and completely dissipative, then all characteristic indicators lie in the lower half-plane of the complex plane. All particular solutions are fading functions, and therefore, the general solution is a fading function of time. If the system has incomplete dissipation, then some of its indicators lie in the left half-plane, while others lie on the imaginary axis. Among the partial solutions, there are periodic solutions that meet non-damping degrees of freedom. If the system has negative dissipation, some of its characteristic indicators may have negative real parts.

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Data Availability

Data available on request due to ethical reasons.

Conflicts of Interest

The authors declare that they have no conflict of interest. "Funders played no role in the design of the study, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results".

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