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On the Solution of an Integro-Differential Equation Using the Fourier Method

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Abstract

A nonhomogeneous integro-differential equation of viscoelasticity with zero boundary conditions is studied. To facilitate analysis, auxiliary functions are introduced, enabling the equation to be rewritten in a form suitable for rigorous investigation. The problem is subsequently reduced to a second-kind Volterra integral equation, which is solved using the Fourier method. A theorem establishing the uniqueness of the solution to the posed problem is also proven.

Keywords: Hyperbolic equation, Integro-differential equation, Fourier method, Grönwall's inequality.

1 | Introduction

Direct and inverse problems for integro-differential equations of hyperbolic type represent a rapidly developing area in modern mathematical physics. Problems of this nature frequently occur across various areas of applied science, including electrodynamics, acoustics, quantum scattering theory, geophysics, astronomy, and related disciplines. These phenomena give rise to integro-differential equations that govern the propagation of elastic and electromagnetic waves in media with memory, where the present state depends not only on the current parameters but also on the entire history of the system. Mathematically, this introduces convolution-type integral terms into the right-hand sides of the classical wave equations, accounting for delay or aftereffect phenomena.

One of the pressing problems in modern mathematical physics is the investigation of the existence and uniqueness of solutions to initial value problems for linear Volterra convolution-type integro-differential equations in Banach spaces. The specificity of such objects lies in their dual nature. Integro-differential equations are functional equations in which the unknown function and its derivatives appear both under the

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integral sign and outside of it. Consequently, unlike integral equations, they also contain derivatives of the unknown function. Similar to integral equations, integro-differential equations are classified as Fredholm or Volterra. In contemporary literature, such technical and natural systems are referred to as systems with aftereffect, heredity, or dynamic memory. Volterra proposed the mathematical description of the governing laws for such systems in a series of articles [1] and a review [2] based on integral and integro-differential equations, which were subsequently named after him and remain relevant to this day.

A series of profound results on the well-posedness of Volterra integro-differential equations of first and second order that are not resolved with respect to the highest-order derivative, along with findings on the spectrum of corresponding operator-valued functions, were established by Kopachevsky and his students [3], [4]. It should be noted that in these works, solvability was examined in spaces of continuously differentiable functions over a finite time interval.

References [5] and [6] present a comprehensive study of different approaches to formulating direct and inverse problems for second-order partial differential equations.

This study addresses the problem of finding a solution to a nonhomogeneous integro-differential equation of viscoelasticity, in which the principal part contains a general hyperbolic operator, taking into account the conditions that define the direct (initial-boundary) problem. Similar issues for homogeneous integro-differential equations were previously examined in [7, 8].

2 | Statement of the Problem

In the domain, $\Omega=\{(x,t):0<x<p, 0<t\leq T\}$ we consider the integro-differential equation.

$$\frac{\partial^2 w}{\partial t^2}-\frac{\partial^2 w}{\partial x^2}=\int_0^t s(t-\theta)\frac{\partial^2 w}{\partial x^2}(x,\theta)d\theta+f(x),\ (x,t)\in\Omega. \tag{1}$$

with initial conditions

$$w\,|_{t=0}=a(x), \left.\frac{\partial w}{\partial t}\right|_{t=0}=\eta(x), 0<x<l. \tag{2}$$

boundary conditions

$$w(0,t)=0,\quad w(p,t)=0, t>0, \tag{3}$$

where $s(t), \varphi(x), \eta(x)$ and $f(x,t)-$ are given functions $p>0.$.

In geophysics, integro-differential equations arise when studying the properties of a medium using seismic waves. Essentially, under smoothness assumptions, the system of equations for the non-elastic Boltzmann model (one of the most general for a linear non-elastic medium) reduces to this equation.

Definition 1 ([9]). A function $u(x,t)\in C^2(\bar{\Omega})$ that satisfies all the equations and conditions of the mixed *Problems (1)–(3)* is called its classical solution.

3 | Preliminary Construction

We introduce a new function $u(x,t)$, defining it by the relation:

$$u(x,t):=\left[w(x,t)+s(t-\theta)w(x,\theta)\ d\theta\right]\exp\left(-m(0)t/2\right). \tag{4}$$

It can readily be verified [10] that the formula is expressed via the function:

$$w(x,t)=\exp(m(0)t/2)u(x,t)+\int_0^t K(t-\theta)\exp(m(0)\theta/2)u(x,\theta)\,d\theta,$$

where

$$K(t)=-s(t)-\int_0^t s(t-\theta)K(\theta)\,d\theta,\ t>0.$$

For simplicity, we assume that $s(0)=K(0).$. Consequently, as follows $s'(0)=K'(0).$, for the new functions $u(x,t)$ and $K(t)$, *Eqs. (1)-(3)* acquire the form:

$$u_{tt}-u_{xx}=\int_0^t k(t-\theta)u(x,\theta)d\theta+f(x). \quad \textbf{(5)}$$

$$u|_{t=0}=a(x), u_t|_{t=0}=\eta(x)-\frac{K(0)}{2}a(x)=:b(x), 0<x<p. \quad \textbf{(6)}$$

$$u(0,t)=0, \quad u(p,t)=0, t>0, \quad \textbf{(7)}$$

where we have introduced the notation $k(t)=K''(t)\ \exp\left(\frac{K(0)}{2}t\right).$

We will seek the solution to *Problems (5)-(7)* in the form of a Fourier series

$$u(x,t)=\sum_{n=1}^{\infty}T_n(t)X_n(x). \quad \textbf{(8)}$$

If we substitute Expression *Eq. (8)* into the given equation, the resulting relation can be rewritten in the form of a spectral problem. In particular, the eigenvalues of this operator are expressed by the functions $\gamma_n=\frac{\pi}{p}n$. In contrast, the eigenfunctions associated with these eigenvalues take the form $X_n(x)=\frac{\sqrt{2}\sin\gamma_n x}{\sqrt{p}}.$ of Fourier coefficients, calculated using the formula, which are numerical parameters that determine the amplitude and phase of the harmonic components of a function.

Let a_n, b_n, f_n be the Fourier coefficients.

$$a_n=\int_0^p a(x)X_n(x)dx, b_n=\int_0^p b(x)X_n(x)dx, f_n=\int_0^p f(x)X_n(x)dx. \quad \textbf{(9)}$$

of the functions $a(x)$, $b(x)$ and $f(x)$:

$$\begin{cases} a(x)=\sum_{n=1}^{\infty}a_nX_n(x), \\ b(x)=\sum_{n=1}^{\infty}b_nX_n(x), \\ f(x)=\sum_{n=1}^{\infty}f_nX_n(x). \end{cases} \quad \textbf{(10)}$$

Remark. In this case, the existence of Fourier series expansions for the functions $a(x)$, $b(x)$, $f(x)$ is assumed.

By expressing *Eqs. (5)* and *(6)* in terms of the variable using *Eq. (8)*, we obtain the following mathematical problem, which constitutes a system of equations with specific conditions, the solution of which allows us to investigate the behavior of the original function.

$$T_n''(t)+\gamma_n^2 T_n(t)+\int_0^t k(t-\theta)T_n(\theta)d\theta = f_n. \tag{11}$$

With initial conditions

$$T_n\big|_{t=0}(0)=a_n,\ \left.\frac{dT_n}{dt}\right|_{t=0}=b_n. \tag{12}$$

The solutions to Problems (11) and (12) can be expressed as corresponding integral equations, providing a convenient framework for further investigation. This formulation not only allows for the determination of the existence and uniqueness of the solution but also facilitates the application of analytical methods for its construction. In particular, expressing the problem in integral form is well-suited to the Fourier method, which decomposes functions into series and reduces the original integro-differential equation to a system of algebraic equations for the expansion coefficients. This approach simplifies the study of the solution's properties and allows both analytical and numerical examination of the system's behavior.

$$T_n(t)=\left(a_n+\frac{f_n}{\gamma_n}\right)\cos\gamma_n t+\frac{1}{\gamma_n}b_n\sin\gamma_n t-\frac{f_n}{\gamma_n}+\frac{1}{\gamma_n}\int_0^t \sin\gamma_n(t-s)ds\int_0^s k(s)T_n(\tau-s)d\tau. \tag{13}$$

For each fixed n , *Eq. (13)* is a Volterra integral equation of the second kind with respect to the function $T_n(t)$. . From the theory of integral equations, it follows that the solution to this equation is unique and can be obtained by the method of successive approximations.

Lemma. When they are sufficiently large, the following inequalities hold:

$$|T_n(t)|\le S_1\left(\left|a_n+\frac{f_n}{\gamma_n}\right|+\frac{1}{\gamma_n}|b_n|+\gamma_n|f_n|\right)\exp\{2T\|k\|\}. \tag{14}$$

$$|T_n''(t)|\le S_2\left(\gamma_n^2\left|a_n+\frac{f_n}{\gamma_n}\right|+\gamma_n|b_n|+\gamma_n\|f_n\|\right)\left(\exp\{2T\|k\|\}+1\right). \tag{15}$$

with the same data a_n, b_n and $f_n(t)$, where $S_1, S_2 -$ are positive constants depending only on T; $\|k\|=\max\limits_{t\in[0,T]}|k(t)|$.

Proof. From *Eq. (13)* it follows that

$$\begin{aligned}|T_n(t)|&=\left|\left(a_n+\frac{f_n}{\gamma_n}\right)\cos\gamma_n t+\gamma_n^{-1}b_n\sin\gamma_n t+\gamma_n^{-1}f_n(t)+\gamma_n^{-1}\int_0^t\sin\lambda_m(t-s)\int_0^s k(s-\theta)A_m(\theta)d\theta ds\right|\le\\ &\left|a_n+\frac{f_n}{\gamma_n}\right|+\gamma_n^{-1}|b_n|+\gamma_n^{-1}\|f_n(t)\|+\gamma_n^{-1}\left|\int_0^t\sin\lambda_n(t-s)\int_0^s k(s-\theta)T_n(\theta)d\theta ds\right|\le\\ &\left|a_n+\frac{f_n}{\gamma_n}\right|+\gamma_n^{-1}|b_n|+\gamma_n^{-1}\|f_n(t)\|+\gamma_n^{-1}\|k\|\left|\int_0^s(t-\theta)T_n(\theta)d\theta ds\right|.\end{aligned} \tag{16}$$

The estimate *Eq. (14)* is obtained directly from *Eq. (16)* by applying the well-known Grönwall inequality. To derive *Eq. (15)*, we perform a formal double differentiation of series *Eq. (13)* with respect to the corresponding variable and repeat a similar procedure. This method allows careful control over the behavior of the derivatives and ensures the correctness of the resulting expressions when taking limits. Moreover, the use of formal differentiation of the series allows one to reveal the internal structure of the solution and determine the influence of each term on the overall dynamics of the function. This approach also provides a foundation for subsequent convergence analysis. By formally differentiating series *Eq. (8)* with respect to the variables x and t, we obtain the corresponding series expansions:

$$u_{tt}(x,t)=\sum_{n=1}^{\infty}T_n''(t)X_n(x). \tag{17}$$

$$u_{xx}(x,t)=-\gamma_n^2\sum_{n=1}^{\infty}T_n(t)X_n(x). \tag{18}$$

For the functional expansion *Eq. (8)*, a majorant series is selected.

$$|u(x,t)|\le h_1\sum_{n=1}^{\infty}\left(|a_n|+\gamma_n^{-1}|b_n|+\left(\gamma_n+\frac{1}{\gamma_n}\right)|f_n|\right). \tag{19}$$

By applying the estimates *Eqs. (14)* and *(15)* to series *Eqs. (17)* and *(18)*, the following inequalities can be established:

$$|u_{tt}(x,t)|\le h_2\sum_{n=1}^{\infty}\left(\gamma_n^2|a_n|+\gamma_n|b_n|+\left(\gamma_n+\frac{1}{\gamma_n}\right)|f_n|\right). \tag{20}$$

$$|u_{xx}(x,t)|\le h_3\sum_{n=1}^{\infty}\left(\gamma_n^2|a_n|+\gamma_n|b_n|+\left(\gamma_n+\frac{1}{\gamma_n}\right)|f_n|\right), \tag{21}$$

where h_1, h_2 and h_3 denote positive constants that depend exclusively on T. *Eqs. (19)–(21)* imply that, for any $(x,t)\in\overline{\Omega}$, the series *Eqs. (8), (17), (18)* are dominated by the series

$$h_4\sum_{n=1}^{\infty}\left(n^2|a_n|+n\left(|b_n|+|f_n|\right)\right), \tag{22}$$

where h_4 − is a constant that depends only on T.

4 | Main Result

The main result of this work is the following theorem.

Theorem 1. Let $s(t)\in C[0,T]$, $a(x)\in C^4[0,p], b(x)\in C^3[0,p]$ и $f(x)\in C^3[0,p]$, and additionally let the following boundary conditions hold:

$$a(0)=a(p)=a''(0)=a''(p)=0.$$
$$b(0)=b(p)=b''(0)=b''(p)=0.$$
$$f(0)=f(p)=f''(0)=f''(p)=0.$$

Then there exists a unique solution to *Eqs. (5)-(7)*.

Proof: By performing a fourfold integration by parts on *Eq. (9)*, while accounting for all conditions specified in the theorem, we obtain a form that incorporates the boundary effects and facilitates the derivation of the necessary estimates. This approach transfers the differential operators onto auxiliary functions, simplifying the subsequent analysis and yielding a final expression suitable for further investigation and for establishing the theorem's claims.

$$\left(\frac{\pi n}{p}\right)^4 a_n = 2\int_0^p \frac{a''''(x)\sin\gamma_n x}{p}dx = a_n^{(4)}.$$

From this, we obtain the following representation for $|a_n|$:

$$|a_n| = \frac{|a_n^{(4)}|}{\left(\frac{\pi n}{p}\right)^4}.$$

Performing a similar procedure for $|b_n|$, $\| f_n(t) \|$, we have

$$|b_n| = \frac{|b_n^{(3)}|}{\left(\frac{\pi n}{p}\right)^3}, \quad |f_n| = \frac{|f_n^{(3)}|}{\left(\frac{\pi n}{p}\right)^3}.$$

From these formulas, using Bessel's inequality, we obtain

$$\sum_{n=1}^{\infty} |a_n^{(4)}|^2 \leq \int_0^p \left(\frac{d^4 a(x)}{dx^4}\right)^2 dx.$$

$$\sum_{n=1}^{\infty} |b_n^{(3)}|^2 \leq \int_0^p \left(\frac{d^3 b(x)}{dx^3}\right)^2 dx.$$

$$\sum_{n=1}^{\infty} \left|f_n^{(3)}\right|^2 \leq \int_0^p \left(\frac{d^3 f(x)}{dx^3}\right)^2 dx.$$

From these relations, the convergence of the series *Eq. (22)* follows. Consequently, the series *Eqs. (8), (18), (19)* converge uniformly. Thus, the function $u(x,t)$ defined by *Eq. (8)* is a solution to *Problems (5)-(7)*. The uniqueness of this solution is established by the usual method. The function $w(x,t)$ is related to the function $u(x,t)$ by a relation. Consequently, the function is a solution to *Problems (1)-(3)*.

The theorem is proven.

Author Contribution

For research articles with multiple authors, provide a short paragraph that identifies each contribution. The following statements should be used: "Conceptualization, J.S. and Sh..F.; Methodology, J..S. and Sh.F.; Software, J..S. and Sh.F.; Validation, J..S. and Sh. formal analysis, J..S. and Sh.F.; investigation, J..S. and Sh.F.; resources, J..S. and Sh.F.; data maintenance, J..S. and Sh.F.; writing-creating the initial design, J..S. and Sh.F.; writing-reviewing and editing, J..S. and Sh.F.; visualization, J..S. and Sh.F.; monitoring, J..S. and Sh.F.; project management, J..S. and Sh.F.; funding procurement, J..S. and Sh.F.; All authors have read and agreed to the published version of the manuscript. Authorship must be limited to those who have made a significant contribution to the work reported. All terms were described in the CT file.

Data Availability

Alldata supporting the reported findings in this research paper are provided within the manuscript.

Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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