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On the Dynamical Problems of Linear Dissipative Mechanical Systems Under Natural Vibrations

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Abstract

This study investigates the dynamic behavior of linear dissipative mechanical systems subjected to natural vibrations, with particular emphasis on the influence of viscoelastic dissipation mechanisms. The research develops mathematical models for analyzing the free vibration response of mechanical systems composed of rigid and deformable bodies, taking into account the hereditary properties of viscoelastic materials. The dissipative effects are represented through additional integral terms appearing in the governing integro-differential equations, where hereditary kernels characterize the time-dependent material behavior and energy dissipation processes. Based on the principles of analytical mechanics and the application of Lagrange's equations, a comprehensive formulation of the equations of motion is established. The resulting mathematical model enables the determination of characteristic exponents, damping coefficients, and eigenfrequencies of the system. To improve the efficiency and accuracy of the computational procedure, modified versions of the Gauss and Muller numerical methods are proposed.

Furthermore, an algorithm employing complex arithmetic is developed for solving the characteristic equations associated with viscoelastic systems and for identifying the corresponding complex eigenvalues. The stability and asymptotic behavior of the oscillatory processes are analyzed through the obtained characteristic roots. Numerical simulations are carried out for representative mechanical systems, and the influence of viscoelastic parameters and hereditary kernels on vibration attenuation, frequency variation, and dynamic stability is investigated. The results demonstrate the effectiveness of the proposed computational approach in accurately predicting the dynamic response of dissipative systems and provide valuable insights into the design and analysis of engineering structures incorporating viscoelastic materials. The developed methodology can be applied to a wide range of engineering problems involving vibration control, structural dynamics, and the assessment of long-term behavior in mechanical systems with energy dissipation mechanisms.

Keywords: Linear dissipative mechanical systems, Natural vibrations, Free oscillations, Damping coefficient, Viscoelasticity, Hereditary kernels, Lagrange equations, Characteristic exponents, Eigenfrequencies, Stability analysis, Asymptotic behavior.

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1 | Introduction

Currently, there are problems with the same and different structures related to dissipative mechanical systems, the solution of which reveals new aspects of oscillations and contributes to the creation of more effective solutions for vibration protection [1–3]. The development of accurate and stable computing technologies plays an important role in solving the problems of durability and stability arising from dynamic impacts in the fields of mechanical engineering, aircraft technology, and ground transport [4], [5]. For structures subjected to periodic external stresses, the use of dynamic vibration dampers is one of the most effective measures. There are also a number of scientific and technical developments on the creation of multi-level and network systems that provide effective dissipation in multi-mode oscillations [6], [7]. In solving problems aimed at reducing dynamic oscillations, theoretical approaches based on structural methods of mathematical modeling [8], [9] are particularly important. In this approach, the mechanical oscillation system is considered as a certain dynamic automatic control system. Thus, the structural diagram constructed for the system serves as a structural equivalent of the initial mathematical model given by differential equations [10], [11]. In the structural model of the protection system, the object can be represented as a dynamic circuit with negative feedback. This approach can lead to the emergence of second-order integrating links. Based on the feedback transfer functions, the system's damping modes are determined [12], [13]. From the point of view of the structural model, negative feedback reflects the elastic reaction of the system relative to the object of protection or conditional stiffness, which changes depending on the frequency of the impact.

2 | Problem Statement and Methodology for Solving It

A linear dissipative mechanical system is represented as a body with volume $V = \sum_{n=1}^{N} V_n$, consisting of N parts with volumes $V_n \left(n = 1; 2; ...; N\right)$, and is bounded by the surface $S = S_p + S_l$. In the part of the surface S_p , in general, external loads are given, and boundary conditions are given on the surface. The contact surfaces meet the conditions of sliding (there is no friction on the contact) or rigidly fixed contact, or there is friction in the contact [14].

On some surfaces of a piecewise-inhomogeneous mechanical system, a concentrated mass L is attached using massless deformable elements (springs).

The relationship between stresses and deformations of massive deformable elements satisfies the following relations:

$$\sigma_{ij}^{(n)} = \tilde{\lambda}_n \theta_n \delta_{ji} + 2\tilde{\mu}_n \varepsilon_{ij}^{(n)}, (n = 1,2,3,\dots,S), \quad (1)$$

where

$$\tilde{\lambda}_n = \frac{v_n \tilde{E}_n}{(1+v_n)(1-2v_n)}; \quad \tilde{\mu}_n = \frac{v_n \tilde{E}_n}{2(1+v_n)}.$$

Here, $\tilde{E}_n$ is the operator modulus of elasticity, which has the following form [2]:

$$\tilde{E}_n(t) = E_{0n}\left[(t) - \int_0^t R_{En}(t-\tau)\,(t)\,d\tau\right], \quad \textbf{(2)}$$

where $\phi(t)$ is an arbitrary time-dependent function;

$R_{En}(t-\tau)$ - relaxation nucleus; E_{0n} is the instantaneous response modulus of elasticity of the material. Now, let's assume that the integral terms expressing the dissipative properties of the viscoelastic body in *Eq. (2)* are very small compared to its instantaneously acting elastic composition. Such conditions allow for the application of the "freezing method" [15]. Then, using this freezing method, the equations in *Eq. (2)* are replaced by approximate expressions of the following type

$$\bar{E}_n\phi = E_{0n}\left[1-\Gamma_n^C(\omega_R) - i\Gamma_n^S(\omega_R)\right]\phi. \quad \textbf{(3)}$$

here

$$\Gamma_n^C(\omega_R) = \int_0^{+\infty} R_n(\tau)\cos\omega_R\tau\,d\tau, \qquad \Gamma_n^S(\omega_R) = \int_0^{+\infty} R_n(\tau)\sin\omega_R\tau\,d\tau.$$

The cosine and sine compositions of the Fourier image of the relaxation nucleus of the material are determined, respectively.

For modeling a viscoelastic substance, the three-parameter relaxation kernel is taken as follows:

$$R_n(t) = A_n e^{-\beta_n t} / t^{1-\alpha_n}.$$

In some special cases, elements belonging to the Lumped Distributed Mass Chain (LDMC) series can be connected to each other through N non-mass deformable connecting elements. For such elements (i.e., elements without mass or having a non-zero volume), the physical dependencies are expressed by the following equations:

$$F_e = -\bar{C}_e\Delta e = -c_e\left[1-\Gamma_e^c(\omega_R) - i\Gamma_e^s(\omega_R)\right]\Delta e,$$

where F_e force in the i-th constituent element, Δe - elongation of this element, ω_R - frequency of external influence.

Some deformable elements can be perfectly elastic; in this case, the nuclei of heredity, characterizing their rheological properties, are equal to zero. If the rheological nuclei of all deformable elements in the system are zero, they exhibit an absolutely elastic property.

Setting of problems in the dynamics of linear dissipative mechanical systems. For the mathematical formulation of the above-mentioned problems, the Euler-Lagrange principle is applied:

$$\delta A = \delta A_\sigma + \delta A_F + \delta A_u + \delta A_{pa} = 0, \quad \textbf{(4)}$$

where $\delta A_\sigma = -\sum_{e=1}^{N}\Gamma_e\delta\Delta l - \sum_{n=1}^{S}\int_{v_n}\sigma_{ij}\delta\varepsilon_{ij}dV,$

$$\delta A_u = -\sum_{k=1}^{L} m_k \frac{d^2\bar{u}}{dt^2}\delta\bar{u}_k - \sum_{n=1}^{S}\int_{V_n}\rho_n \frac{\partial^2\bar{u}}{\partial t^2}\delta\bar{u}dV - \sum_{k=1}^{L} I_k \frac{d^2\bar{u}}{dt^2}\delta\bar{\phi}_k,$$

$$\delta A_F = \sum_{n=1}^{S_1}\int \bar{f}\delta\bar{u}dV - \sum_{n=1}^{S}\int \rho_n \bar{f}\delta\bar{u}dV + \sum_{n=1}^{N_1}\int \bar{F}_n\delta\bar{u}_n + \sum_{k=1}^{N_1}\int \bar{M}_k\delta\bar{\phi}_k.$$

When considering natural oscillations, the *Right Sides (1)-(4)* are identically equal to zero. We will look for the solution in the form:

$q_j = A_j e^{-i\omega t}$, j= 1,…,N,

where $\omega = \omega_R + i\omega_I$ – complex natural frequency. The problem reduces to a complex algebraic problem with eigenvalues of the form:

$$\sum_{k=1}^{N}(A_k(C_{jk}(\omega_R) - \omega^2 a_{jk})) = 0 \quad j = 1, 2, ..., N. \quad \textbf{(5)}$$

with a nonlinear incoming complex parameter. The characteristic equation of *Eq. (5)* is solved numerically using the Muller method. As an initial approximation, a solution close to (5) is chosen for the corresponding conservative problem. In this case, the determinant of the *System (5)* on each iteration of the Muller method is calculated using the Gauss method, with the main element selected by rows and columns. Thus, solving the problem of natural oscillations using the Muller method does not require the disclosure of its determinant.

The solution to the problem of forced oscillations of the *System (5)* will be sought in the form:

$$\sum_{k=1}^{N}(A_k(C_{jk}(\omega_R) - \omega^2 a_{jk})) = 0 \quad j = 1, 2, ..., N, \quad \textbf{(6)}$$

where A_j is the sought complex amplitudes. The problem of stable forced oscillations reduces to a system of non-uniform algebraic equations:

$$\sum_{m=1}^{N}(C_{jm}(\lambda) - \lambda^2 a_{jm})A_m = f_j + \eta_j(q_э),$$

whose solution is carried out by the Gauss method. The result of solving the forced oscillation problem is obtaining the Amplitude-Frequency Characteristics (AFC) of the mechanical system. If a mechanical system has one degree of freedom, then the equation of motion of the system is written as:

$$y'' + \omega^2 y - \int_{-\infty}^{t} R(t - S_1)y(S_1)dS_1 - C\sin pt = -\eta_1(q_э)\sin pt, \quad \textbf{(7)}$$

where p is the frequency of external influence. The partial solution of *Eq. (7)* has the form:

$d_1 \cos p_1 t + d_2 \sin p_1 t = y(t),$

where

$$d_1 = \frac{(-c + \eta_1(q_э))\omega^2\Gamma_S(p_1)}{\left[\omega^2(1 - \Gamma_C(p_1)) - p_1^{\,2}\right]^2 + \omega^4\Gamma_S^2(p_1)}.$$

$$d_2 = \frac{(c - \eta_1(q_э))\left[\omega^2(1 - \Gamma_C(p_1)) - p_1^{\,2}\right]}{\left[\omega^2(1 - \Gamma_C(p_1)) - p_1^{\,2}\right]^2 + \omega^4\Gamma_S^2(p_1)}.$$

$$F_C(p_1)=\int_0^\infty R(\tau)\cos p_1\tau d\tau;\ F_S(p_1)=\int_0^\infty R(\tau)\sin p_1\tau d\tau.$$

When the viscous properties of deformable elements are taken into account through viscous friction, then in matrix form, relative to the matrix - column $\{X\}=\text{colon}(x_1,x_2,...,x_n)$ of the Integro-Differential *Eq. (7)* (IDU), take the form

$$[M]\{\ddot{X}\}+[C]\{\dot{X}\}-\int_{-\infty}^{t}[R(t-\tau)]\{X(\tau)\}d\tau+[K]\{X\}=\{f\}+\{\eta(q_э)\}, \quad \textbf{(8)}$$

where [M] is a positively defined matrix, the elements of which represent concentrated masses, [C] is the matrix of the damping coefficients of deformable bodies, and [K] is the matrix of the elements of the skeletal characteristics of deformable bodies (symmetric square matrix), $[R(t-\tau)]$ is the viscosity matrix without mass elements. The perturbing force is denoted by the column vector -$\{f\}$.

The above matrices have a physical meaning, i.e., M_{jk}, C_{jk} and K_{jk} are, respectively, elements of the mass, attenuation, and stiffness matrices. All matrices are square, j is the number of rows, and k is the number of columns. The elements of this *Eq. (8)* are obtained from the system of differential *Eq. (7)*.

Equation of vibration shielding systems with a finite number of possible movements in directions. The equations of motion of the mathematical model of machines and equipment can be written by the system:

$$[M]\{\ddot{\phi}\}+[C]\{\phi\}+[B]\{\dot{\phi}\}=\{F(t)\}. \quad \textbf{(9)}$$

If active elements are introduced into the mathematical model, then the mathematical model can be written in normal form by matrix differential equations of the form:

$$\{\dot{x}\}=\{F(t)\}+[A]\{x\}, \quad \textbf{(10)}$$

where x is the state vector, and $[A]$ is an arbitrary n-dimensional square, complex matrix. If the system is complex, i.e., the system is not represented as a chain of bodies, then matrices $[B]$ and $[C]$ can be absolutely "dense."

Server communication reactions can be determined using structural diagrams equivalent to automatic control systems. This approach can also be applied to systems where mass-inertial elements are represented as solids [11].

Service communication can be implemented by mechanisms that produce various generalized forces, such as mechanical, electromechanical, hydraulic, electrohydraulic, etc. [12]. Let a linear dissipative mechanical system consist of rigid bodies connected by massless deformable elements. Then *Eq. (5)* reduces to a system of linear differential equations with complex rigidity:

$$\sum_{k=1}^{6N}(a_{jk}\ddot{q}_k+\tilde{C}_{jk}q_k)=f_je^{-i\lambda t}+\delta_{\omega_R,\lambda}\cdot F^{(2)}(\nu)\cdot e^{-i\lambda t},\ (j=1,2,...,6N, \quad \textbf{(11)}$$

where $\lambda-$ frequency; $a_{jk}-$ component of the real matrix of generalized masses; C - generalized stiffness, $\delta_{\omega_R,\lambda}$ - Kroneker symbols; $F^{(2)}(\nu)$ - ponderomotive forces. For natural oscillations, we seek the solution in the form:

$$q_j=A_je^{-i\omega t},\ (j=1,...,6N).$$

Then, substituting the last expression, we obtain:

$$\sum_{\kappa=1}^{6N}(\tilde{C}_{jk}-\omega^2 a_{jk})\cdot A_k=0\ (j=1,...,6N). \tag{12}$$

The characteristic *Eq. (6)* can be solved numerically, for example, using the Muller method. For the initial value, a solution close to the corresponding conservative problem is chosen.

For *System (12)*, we seek the solution in the form:

$$q_j=A_j e^{-i\lambda t},\ (j=1,...,6N), \tag{13}$$

where A_j - unknown amplitudes. Then we get:

$$\sum_{m=1}^{6N}(\tilde{C}_{jm}-\lambda^2 a_{jm})A_m=f_j+\delta_{\omega_R,\lambda}\cdot F^{(2)}(\nu).$$

For systems with deformable bodies, *Eq. (6)* reduces to the system:

$$\sum_{k=1}^{S_2}L_{jk}W_k\pm\rho_j\frac{\partial^2 W_j}{\partial t^2}=\pm\rho f_{1j}e^{i\lambda t}\mp\delta_{\omega_R,\lambda}f_{2j}e^{i\lambda t}, \tag{14}$$

where W_j - component of displacement; ρ_j - density; ρ_j -differentiation operators. The *System (14)* does not include the masses and moments of inertia of non-deformable bodies and the stiffness components of massless deformable elements. These parameters in the proposed approach are considered at the internal points of the j-th rod.

3 | Conclusion

Two main parameters are proposed to express the overall degree of dissipation: the lowest damping rate of natural oscillations and the highest resonance amplitude. The boundary conditions for the *System (14)* are generally not written due to the extreme diversity. Only when posing problems for specific mechanical systems of equations are boundary conditions automatically obtained in the process of integration by parts of the integral terms.

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Conflict of Interest

The authors declare that they have no conflict of interest. "Funders played no role in the design of the study, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results."

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Ethics Approval and Consent to Participate

This article does not contain any studies with human participants performed by the author.

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