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On Estimation of Residual Stresses in Metals After Heat Treatment

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Abstract

Residual stresses developed during heat treatment have a significant effect on the mechanical performance, dimensional accuracy, and service life of metallic components. Therefore, the estimation of these stresses is an important issue in the design and optimization of heat treatment processes. In this paper, a mathematical model for estimating residual stresses in metals after heat treatment is proposed. The transient temperature distribution within the material is determined by solving the second Fourier heat conduction equation. Based on the obtained temperature field, an analytical approach is developed to evaluate the evolution of residual stresses during the heat treatment process. The proposed methodology provides a simple and efficient framework for estimating residual stresses and investigating the influence of thermal conditions on their development. The presented approach can be used as a basis for further studies on the optimization and control of heat treatment processes in metallic materials.

Keywords: Heat treatment of metals, Residual stresses in metals, Analytical approach for analysis.

1 | Introduction

The heat treatment is one of the main stages of the production of metal devices. The quality and resource of these devices depend on their correct implementation. The main factors that ensure the quality of heat treatment are temperature and time. Thus, the above properties of these devices are determined by the cooling rate of the metal [1–4]. There are different approaches that give the possibility to control the cooling rate of metal products, which were elaborated. The manufacturing of different structures in a metallic material is accompanied by the redistribution of various structural defects (atoms, dislocations, ...) [1], [2], [4], [5]. The final fixed structural state of the metal is characterized by an internal stress consisting of the interaction of elastic fields of lattice stresses and defects. Residual stresses arising during heat treatment, combined with workloads, have a negative impact on operational reliability and service life. The main aim of the present

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paper is the formulation and analysis of a model of changes of stress in metals during heat treatment. The accompanying aim of the present paper is to elaborate on an analytical approach to analyze the considered model.

2 | Method of Solution

To solve the considered aim, we determine the distribution of temperature T (x,y,z,t) in space and time by solving the second Fourier's law [6], [7].

$$c\,(T)\frac{\partial T(x,y,z,t)}{\partial t}=\frac{\partial}{\partial x}\left[\lambda(x,y,z,T)\frac{\partial T(x,y,z,t)}{\partial x}\right]+\frac{\partial}{\partial y}\left[\lambda(x,y,z,T)\frac{\partial T(x,y,z,t)}{\partial y}\right]+$$
$$\frac{\partial}{\partial z}\left[\lambda(x,y,z,T)\frac{\partial T(x,y,z,t)}{\partial z}\right]+p(x,y,z,t). \tag{1}$$

with boundary and initial conditions

$$\left.\frac{\partial T(x,y,z,t)}{\partial x}\right|_{x=0}=0;\ \left.\frac{\partial T(x,y,z,t)}{\partial x}\right|_{x=L_x}=0;\ \left.\frac{\partial T(x,y,z,t)}{\partial y}\right|_{y=0}=0;\ \left.\frac{\partial T(x,y,z,t)}{\partial y}\right|_{y=L_y}=0;$$
$$\left.\frac{\partial T(x,y,z,t)}{\partial z}\right|_{z=0}=0;\ \left.\frac{\partial T(x,y,z,t)}{\partial z}\right|_{z=L_z}=0;\ T\,(x,y,z,0)\ =Tr. \tag{2}$$

Here λ is the coefficient of thermal conductivity. The value of the coefficient depends on the properties of the considered materials and temperature. Dependence of the coefficient of thermal conductivity could be approximated by the following function: $\lambda(x,y,z,T)=\lambda_{ass}(x,y,z)[1+\mu\,T_d^{\varphi}/T^{\varphi}(x,y,z,t)]$. $c\,(T)=c_{ass}[1-\vartheta\exp(-T(x,y,z,t)/T_d)]$ is the heat capacitance; T_d is the Debye temperature. In the case when the current temperature T (x,y,z, t) is equal to or larger than the Debye temperature T_d, then we can use the following approximation $c\,(T)\approx c_{ass}$. This issue is of most interest for our case. p (x,y,z,t) is the volumetric density of power, which is released in the considered sample. First of all, we estimate the distribution of temperature. To make the estimation, we solve *Eq. (1)* by the method of averaging of function corrections [8–10]. In the framework of this method, we replace the distribution of temperature in space and time by an unknown average value α_{1T} on the right side of *Eq. (1)*. The replacement gives a possibility to obtain the following equation to determine the first-order approximation of the considered temperature.

$$c_{ass}\frac{\partial\,T_1(x,y,z,t)}{\partial t}=p(x,y,z,t). \tag{3}$$

Integration of the left and right sides of *Eq. (3)* in time gives a possibility to obtain the required approximation in the following form.

$$T_1(x,y,z,t)=\frac{1}{c_{ass}}\int_0^t p\,(x,y,z,\tau)\,d\tau+T_r. \tag{4}$$

Not yet known average value α_{1T} was calculated by the following standard relation [8–10].

$$\alpha_{1T}=\frac{1}{\Theta L_x L_y L_z}\int_0^{\Theta}\int_0^{L_x}\int_0^{L_y}\int_0^{L_z}T_1(x,y,z,t)\,dz\,dy\,dx\,dt. \tag{5}$$

The second-order approximation of temperature was calculated by standard replacing of the required function T (x,y,z,t) on sum of average value of the considered approximation and approximation with the previous order (i.e. T (x,y,z,t)→$\alpha_{T2}+T_1$(x,y,z, t)) in the right side of *Eq. (1)*. The replacement gives a possibility to obtain equation to calculate the considered second-order approximation in the following form.

$$\begin{gathered}
+\frac{\partial}{\partial y}\left(\lambda_{ass}(x,y,z)\left\{1+\frac{\mu\, T_d^{\varphi}}{\left[\alpha_{2T}+T_1(x,y,z,t)\right]^{\varphi}}\right\}\frac{\partial\, T_1(x,y,z,t)}{\partial y}\right)+\\
c_{ass}\frac{\partial\, T_2(x,y,z,t)}{\partial t}=\frac{\partial}{\partial x}\left(\lambda_{ass}(x,y,z)\left\{1+\frac{\mu\, T_d^{\varphi}}{\left[\alpha_{2T}+T_1(x,y,z,t)\right]^{\varphi}}\right\}\frac{\partial\, T_1(x,y,z,t)}{\partial x}\right)+\\
+\frac{\partial}{\partial z}\left(\lambda_{ass}(x,y,z)\left\{1+\frac{\mu\, T_d^{\varphi}}{\left[\alpha_{2T}+T_1(x,y,z,t)\right]^{\varphi}}\right\}\frac{\partial\, T_1(x,y,z,t)}{\partial z}\right)+p\,(x,y,z,t)\cdot
\end{gathered}\tag{6}$$

Termwise differentiation gives a possibility to transform the equation to the following form.

$$\begin{gathered}
c_{ass}\frac{\partial\, T_2(x,y,z,t)}{\partial\, t}=\left\{1+\frac{\mu\, T_d^{\varphi}}{\left[\alpha_{2T}+T_1(x,y,z,t)\right]^{\varphi}}\right\}\frac{\partial\, T_1(x,y,z,t)}{\partial x}\frac{\partial\,\lambda_{ass}(x,y,z)}{\partial x}-\lambda_{ass}(x,y,z)\times\\
\times\varphi\,\mu\, T_d^{\varphi}\left[\frac{\partial\, T_1(x,y,z,t)}{\partial x}\right]^2\frac{1}{\left[\alpha_{2T}+T_1(x,y,z,t)\right]^{\varphi+1}}+\lambda_{ass}(x,y,z)\left\{1+\frac{\mu\, T_d^{\varphi}}{\left[\alpha_{2T}+T_1(x,y,z,t)\right]^{\varphi}}\right\}\times\\
\times\frac{\partial^2 T_1(x,y,z,t)}{\partial x^2}+\left\{1+\frac{\mu\, T_d^{\varphi}}{\left[\alpha_{2T}+T_1(x,y,z,t)\right]^{\varphi}}\right\}\frac{\partial\, T_1(x,y,z,t)}{\partial y}\frac{\partial\,\lambda_{ass}(x,y,z)}{\partial y}-\left[\frac{\partial\, T_1(x,y,z,t)}{\partial y}\right]^2\times\\
\times\frac{\varphi\,\mu\, T_d^{\varphi}\,\lambda_{ass}(x,y,z)}{\left[\alpha_{2T}+T_1(x,y,z,t)\right]^{\varphi+1}}+\lambda_{ass}(x,y,z)\left\{1+\frac{\mu\, T_d^{\varphi}}{\left[\alpha_{2T}+T_1(x,y,z,t)\right]^{\varphi}}\right\}\frac{\partial^2 T_1(x,y,z,t)}{\partial y^2}+\\
+\frac{\partial\, T_1(x,y,z,t)}{\partial z}\frac{\partial\,\lambda_{ass}(x,y,z)}{\partial z}\left\{1+\frac{\mu\, T_d^{\varphi}}{\left[\alpha_{2T}+T_1(x,y,z,t)\right]^{\varphi}}\right\}-\frac{\varphi\,\mu\, T_d^{\varphi}\lambda_{ass}(x,y,z)}{\left[\alpha_{2T}+T_1(x,y,z,t)\right]^{\varphi+1}}\times\\
\times\left[\frac{\partial\, T_1(x,y,z,t)}{\partial z}\right]^2+\lambda_{ass}(x,y,z)\left\{1+\frac{\mu\, T_d^{\varphi}}{\left[\alpha_{2T}+T_1(x,y,z,t)\right]^{\varphi}}\right\}\frac{\partial^2 T_1(x,y,z,t)}{\partial z^2}+p\,(x,y,z,t)\cdot
\end{gathered}\tag{6a}$$

Integration of the above equation over time gives a possibility to obtain an equation to determine the second-order approximation of temperature in the following form.

$$\begin{gathered}
c_{ass}\,T_2(x,y,z,t)=\frac{\partial\,\lambda_{ass}(x,y,z)}{\partial x}\int_0^t\left\{1+\frac{\mu\, T_d^{\varphi}}{\left[\alpha_{2T}+T_1(x,y,z,\tau)\right]^{\varphi}}\right\}\frac{\partial\, T_1(x,y,z,\tau)}{\partial x}d\,\tau-\\
-\varphi\,\mu\, T_d^{\varphi}\int_0^t\left[\frac{\partial\, T_1(x,y,z,\tau)}{\partial x}\right]^2\frac{\lambda_{ass}(x,y,z)\,d\,\tau}{\left[\alpha_{2T}+T_1(x,y,z,\tau)\right]^{\varphi+1}}+\lambda_{ass}(x,y,z)\int_0^t\frac{\partial^2 T_1(x,y,z,\tau)}{\partial x^2}\times\\
\times\left\{1+\frac{\mu\, T_d^{\varphi}}{\left[\alpha_{2T}+T_1(x,y,z,\tau)\right]^{\varphi}}\right\}d\,\tau+\int_0^t\left\{1+\frac{\mu\, T_d^{\varphi}}{\left[\alpha_{2T}+T_1(x,y,z,\tau)\right]^{\varphi}}\right\}\frac{\partial\, T_1(x,y,z,\tau)}{\partial y}d\,\tau\times\\
\times\frac{\partial\,\lambda_{ass}(x,y,z)}{\partial y}-\int_0^t\left[\frac{\partial\, T_1(x,y,z,\tau)}{\partial y}\right]^2\frac{\varphi\,\mu\, T_d^{\varphi}\,\lambda_{ass}(x,y,z)}{\left[\alpha_{2T}+T_1(x,y,z,\tau)\right]^{\varphi+1}}d\,\tau+\lambda_{ass}(x,y,z)\times\\
\times\int_0^t\left\{1+\frac{\mu\, T_d^{\varphi}}{\left[\alpha_{2T}+T_1(x,y,z,\tau)\right]^{\varphi}}\right\}\frac{\partial^2 T_1(x,y,z,\tau)}{\partial y^2}d\,\tau+\int_0^t\left\{1+\frac{\mu\, T_d^{\varphi}}{\left[\alpha_{2T}+T_1(x,y,z,\tau)\right]^{\varphi}}\right\}\times
\end{gathered}\tag{6b}$$

$$\begin{gathered}
\times \frac{\partial T_1(x,y,z,\tau)}{\partial z} d\tau \frac{\partial \lambda_{ass}(x,y,z)}{\partial z} - \varphi \mu \int_0^t \left[\frac{\partial T_1(x,y,z,\tau)}{\partial z} \right]^2 \frac{1}{[\alpha_{2T} + T_1(x,y,z,\tau)]^{\varphi+1}} d\tau \times \\
\times T_d^{\varphi} \lambda_{ass}(x,y,z) + \int_0^t p(x,y,z,\tau) d\tau + \int_0^t \left\{ 1 + \frac{\mu T_d^{\varphi}}{[\alpha_{2T} + T_1(x,y,z,\tau)]^{\varphi}} \right\} \frac{\partial^2 T_1(x,y,z,\tau)}{\partial z^2} d\tau \times \\
\times \lambda_{ass}(x,y,z).
\end{gathered}$$

Average value α_{T2} was calculated by using the following standard relation [8–10].

$$\alpha_{2T} = \frac{1}{\Theta L_x L_y L_z} \int_0^{\Theta} \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \left[T_2(x,y,z,t) - T_1(x,y,z,t) \right] dz\,dy\,dx\,dt. \tag{7}$$

Substitution of the first- and the second-order approximations of the considered temperature gives a possibility to obtain the following equation to determine the average value α_{T2}.

$$\begin{gathered}
\int_0^{\Theta} (\Theta - t) \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \frac{\partial \lambda_{ass}(x,y,z)}{\partial x} \left\{ 1 + \frac{\mu T_d^{\varphi}}{\left[\alpha_{2T} + T_1(x,y,z,t)\right]^{\varphi}} \right\} \frac{\partial T_1(x,y,z,t)}{\partial x} dz\,dy\,dx\,dt - \varphi \mu \times \\
T_d^{\varphi} \int_0^{\Theta} (\Theta - t) \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \left[\frac{\partial T_1(x,y,z,t)}{\partial x} \right]^2 \frac{\lambda_{ass}(x,y,z)\,dz\,dy\,dx\,dt}{\left[\alpha_{2T} + T_1(x,y,z,t)\right]^{\varphi+1}} + \int_0^{\Theta} (\Theta - t) \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \lambda_{ass}(x,y,z) \times \\
\left\{ 1 + \frac{\mu T_d^{\varphi}}{\left[\alpha_{2T} + T_1(x,y,z,t)\right]^{\varphi}} \right\} \frac{\partial^2 T_1(x,y,z,t)}{\partial x^2} dz\,dy\,dx\,dt + \int_0^{\Theta} (\Theta - t) \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \frac{\partial \lambda_{ass}(x,y,z)}{\partial y} \times \\
\left\{ 1 + \frac{\mu T_d^{\varphi}}{\left[\alpha_{2T} + T_1(x,y,z,t)\right]^{\varphi}} \right\} \frac{\partial T_1(x,y,z,t)}{\partial y} dz\,dy\,dx\,dt - \varphi \mu\, T_d^{\varphi} \int_0^{\Theta} \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \left[\frac{\partial T_1(x,y,z,t)}{\partial y} \right]^2 \times \\
\frac{\lambda_{ass}(x,y,z)\,dz\,dy\,dx}{\left[\alpha_{2T} + T_1(x,y,z,\tau)\right]^{\varphi+1}} (\Theta - t) dt + \int_0^{\Theta} (\Theta - t) \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \lambda_{ass}(x,y,z) \left\{ 1 + \frac{\mu T_d^{\varphi}}{\left[\alpha_{2T} + T_1(x,y,z,t)\right]^{\varphi}} \right\} \times \\
\frac{\partial^2 T_1(x,y,z,t)}{\partial y^2} dz\,dy\,dx\,dt + \int_0^{\Theta} (\Theta - t) \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \left\{ 1 + \frac{\mu T_d^{\varphi}}{\left[\alpha_{2T} + T_1(x,y,z,t)\right]^{\varphi}} \right\} \frac{\partial T_1(x,y,z,t)}{\partial z} \times \\
\frac{\partial \lambda_{ass}(x,y,z)}{\partial z} dz\,dy\,dx\,dt - \int_0^{\Theta} (\Theta - t) \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \left[\frac{\partial T_1(x,y,z,t)}{\partial z} \right]^2 \frac{\lambda_{ass}(x,y,z)\,dz\,dy\,dx}{\left[\alpha_{2T} + T_1(x,y,z,t)\right]^{\varphi+1}} dt \times \\
\varphi \mu\, T_d^{\varphi} + \int_0^{\Theta} (\Theta - t) \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} p(x,y,z,t)\, dz\,dy\,dx\,dt + \int_0^{\Theta} \int_0^{L_x} \int_0^{L_y} \int_0^{L_z} \left\{ 1 + \frac{\mu T_d^{\varphi}}{\left[\alpha_{2T} + T_1(x,y,z,t)\right]^{\varphi}} \right\} \times \\
\times \lambda_{ass}(x,y,z) \frac{\partial^2 T_1(x,y,z,t)}{\partial z^2} dz\,dy\,dx (\Theta - t)\, dt = 0.
\end{gathered} \tag{8}$$

Solution of the above equation (i.e., average value α_{T2}) depends on the value of the parameter φ. The solution of the above equation could be obtained by standard approaches [11]. Stress in metals was estimated by solving the following system of equations [12].

$$\begin{cases} \rho\left(x,y,z,t\right)\dfrac{\partial^2 u_x\left(x,y,z,t\right)}{\partial t^2}=\dfrac{\partial \sigma_{xx}\left(x,y,z,t\right)}{\partial x}+\dfrac{\partial \sigma_{xy}\left(x,y,z,t\right)}{\partial y}+\dfrac{\partial \sigma_{xz}\left(x,y,z,t\right)}{\partial z} \\ \rho\left(x,y,z,t\right)\dfrac{\partial^2 u_y\left(x,y,z,t\right)}{\partial t^2}=\dfrac{\partial \sigma_{yx}\left(x,y,z,t\right)}{\partial x}+\dfrac{\partial \sigma_{yy}\left(x,y,z,t\right)}{\partial y}+\dfrac{\partial \sigma_{yz}\left(x,y,z,t\right)}{\partial z}. \\ \rho\left(x,y,z,t\right)\dfrac{\partial^2 u_z\left(x,y,z,t\right)}{\partial t^2}=\dfrac{\partial \sigma_{zx}\left(x,y,z,t\right)}{\partial x}+\dfrac{\partial \sigma_{zy}\left(x,y,z,t\right)}{\partial y}+\dfrac{\partial \sigma_{zz}\left(x,y,z,t\right)}{\partial z} \end{cases} \quad \textbf{(9)}$$

Here σ_{ij} is the stress tensor, which equal to $\sigma_{ij}=K\frac{\partial u_k\left(x,y,z,t\right)}{\partial x_k}\delta_{ij}+\frac{E}{2\left(1+\gamma\right)}\times$ $\times\left[\frac{\partial u_i\left(x,y,z,t\right)}{\partial x_j}+\frac{\partial u_j\left(x,y,z,t\right)}{\partial x_i}-\frac{\delta_{ij}}{3}\frac{\partial u_k\left(x,y,z,t\right)}{\partial x_k}\right]-K\beta\left[T\left(x,y,z,t\right)-T_r\right]$; ρ is the density of metal; δ_{ij} is the Kronecker symbol; E is the tensile modulus (Young modulus); u_i, u_j are the components $u_x(x,y,z,t)$, $u_y(x,y,z,t)$ and $u_z(x,y,z,t)$ of the displacement vector $\vec{u}\left(x,y,z,t\right)$; x_i, x_j are the coordinate x, y, z; γ is Poisson coefficient; $\varepsilon_0 = (a_s-a_{EL})/a_{EL}$ is the mismatch parameter; a_s, a_{EL} are lattice distances in different areas of the considered material; K is the modulus of uniform compression; β is the coefficient of thermal expansion; T_r is the equilibrium temperature, which coincide (for our case) with room temperature. With account of the above relation for the stress tensor, the system of *Eq. (9)* could be written as

$$\rho\frac{\partial^2 u_x\left(x,y,z,t\right)}{\partial t^2}=\left[K+\frac{5E}{6\left(1+\gamma\right)}\right]\frac{\partial^2 u_x\left(x,y,z,t\right)}{\partial x^2}+\left[K-\frac{E}{3\left(1+\gamma\right)}\right]\frac{\partial^2 u_y\left(x,y,z,t\right)}{\partial x\partial y}+$$
$$\frac{E}{2\left(1+\gamma\right)}\left[\frac{\partial^2 u_y\left(x,y,z,t\right)}{\partial y^2}+\frac{\partial^2 u_z\left(x,y,z,t\right)}{\partial z^2}\right]+\left[K+\frac{E}{3\left(1+\gamma\right)}\right]\frac{\partial^2 u_z\left(x,y,z,t\right)}{\partial x\partial z}-\beta K\frac{\partial T\left(x,y,z,t\right)}{\partial x}$$
$$\rho\frac{\partial^2 u_y\left(x,y,z,t\right)}{\partial t^2}=\frac{E}{2\left(1+\gamma\right)}\left[\frac{\partial^2 u_y\left(x,y,z,t\right)}{\partial x^2}+\frac{\partial^2 u_x\left(x,y,z,t\right)}{\partial x\partial y}\right]-K\beta\frac{\partial T\left(x,y,z,t\right)}{\partial y}+$$
$$\left[\frac{5E}{12\left(1+\gamma\right)}+K\right]\frac{\partial^2 u_y\left(x,y,z,t\right)}{\partial y^2}+\frac{\partial}{\partial z}\left\{\frac{E}{2\left(1+\gamma\right)}\left[\frac{\partial u_y\left(x,y,z,t\right)}{\partial z}+\frac{\partial u_z\left(x,y,z,t\right)}{\partial y}\right]\right\}+$$
$$\left[K-\frac{E}{6\left(1+\gamma\right)}\right]\frac{\partial^2 u_y\left(x,y,z,t\right)}{\partial y\partial z}+K\frac{\partial^2 u_y\left(x,y,z,t\right)}{\partial x\partial y}, \quad \textbf{(10)}$$
$$\rho\frac{\partial^2 u_z\left(x,y,z,t\right)}{\partial t^2}=\left[\frac{\partial^2 u_z\left(x,y,z,t\right)}{\partial x^2}+\frac{\partial^2 u_z\left(x,y,z,t\right)}{\partial y^2}+\frac{\partial^2 u_x\left(x,y,z,t\right)}{\partial x\partial z}+\frac{\partial^2 u_y\left(x,y,z,t\right)}{\partial y\partial z}\right]\times$$
$$\frac{E}{2\left(1+\sigma\right)}+\frac{\partial}{\partial z}\left\{K\left[\frac{\partial u_x\left(x,y,z,t\right)}{\partial x}+\frac{\partial u_y\left(x,y,z,t\right)}{\partial y}+\frac{\partial u_x\left(x,y,z,t\right)}{\partial z}\right]\right\}+\frac{1}{6}\frac{\partial}{\partial z}\left\{\frac{E}{1+\sigma}\times\right.$$
$$\left.\left[6\frac{\partial u_z\left(x,y,z,t\right)}{\partial z}-\frac{\partial u_x\left(x,y,z,t\right)}{\partial x}-\frac{\partial u_y\left(x,y,z,t\right)}{\partial y}-\frac{\partial u_z\left(x,y,z,t\right)}{\partial z}\right]\right\}-K\beta\frac{\partial T\left(x,y,z,t\right)}{\partial z}.$$

Conditions for the above equations could be written as

$$\frac{\partial\,\vec{u}\left(0,y,z,t\right)}{\partial x}=0;\ \frac{\partial\vec{u}\left(L_x,y,z,t\right)}{\partial x}=0;\ \frac{\partial\,\vec{u}\left(x,0,z,t\right)}{\partial y}=0;\ \frac{\partial\,\vec{u}\left(x,L_y,z,t\right)}{\partial y}=0;\ \frac{\partial\,\vec{u}\left(x,y,0,t\right)}{\partial z}=0;$$
$$\frac{\partial\,\vec{u}\left(x,y,L_z,t\right)}{\partial z}=0;\ \vec{u}\left(x,y,z,0\right)=\vec{u}_0;\ \vec{u}\left(x,y,z,\infty\right)=\vec{u}_0.$$

Now we determine solutions of *Eq. (10)*. To determine the first-order approximations of the considered components in the framework of the method of averaging of functions corrections [8–10], we substitute not yet known average values α_i of the considered components instead of the required functions. The substitution gives a possibility to obtain the following equations to determine the considered approximations.

$$\rho\frac{\partial^2 u_{1x}(x,y,z,t)}{\partial t^2}=-K\beta\frac{\partial T(x,y,z,t)}{\partial x},\ \rho\frac{\partial^2 u_{1y}(x,y,z,t)}{\partial t^2}=-K\beta\frac{\partial T(x,y,z,t)}{\partial y},$$

$$\rho\frac{\partial^2 u_{1z}(x,y,z,t)}{\partial t^2}=-K\beta\frac{\partial T(x,y,z,t)}{\partial z}.$$

Integration of the left and right sides of the above equations at time t leads to the following results.

$$u_{1x}(x,y,z,t)=\int_0^t\int_0^\vartheta\frac{K\beta}{\rho}\frac{\partial T(x,y,z,\tau)}{\partial x}\,d\tau\,d\vartheta-\int_0^\infty\int_0^\vartheta\frac{K\beta}{\rho}\frac{\partial T(x,y,z,\tau)}{\partial x}\,d\tau\,d\vartheta+u_{0x}$$

$$u_{1y}(x,y,z,t)=\int_0^t\int_0^\vartheta\frac{K\beta}{\rho}\frac{\partial T(x,y,z,\tau)}{\partial y}\,d\tau\,d\vartheta-\int_0^\infty\int_0^\vartheta\frac{K\beta}{\rho}\frac{\partial T(x,y,z,\tau)}{\partial y}\,d\tau\,d\vartheta+u_{0y}$$

$$u_{1z}(x,y,z,t)=\int_0^t\int_0^\vartheta\frac{K\beta}{\rho}\frac{\partial T(x,y,z,\tau)}{\partial z}\,d\tau\,d\vartheta-\int_0^\infty\int_0^\vartheta\frac{K\beta}{\rho}\frac{\partial T(x,y,z,\tau)}{\partial z}\,d\tau\,d\vartheta+u_{0z}$$

Approximations of the considered components of the displacement vector with the second order and approximations with higher order could be calculated by standard replacement of the required functions in *Eq. (8)* on the following sum $\alpha_i+u_{i-1}(x, y,z,t)$ [8–10]. The replacement gives a possibility to obtain equations to calculate the second-order approximations in the following form.

$$\rho\frac{\partial^2 u_{2x}(x,y,z,t)}{\partial t^2}=\left[K+\frac{5E}{6(1+\gamma)}\right]\frac{\partial^2 u_{1x}(x,y,z,t)}{\partial x^2}+\left[K(z)-\frac{E}{3(1+\gamma)}\right]\frac{\partial^2 u_{1y}(x,y,z,t)}{\partial x\partial y}+$$

$$\left[K+\frac{E}{3(1+\gamma)}\right]\frac{\partial^2 u_{1z}(x,y,z,t)}{\partial x\partial z}+\frac{E}{2(1+\gamma)}\left[\frac{\partial^2 u_{1y}(x,y,z,t)}{\partial y^2}+\frac{\partial^2 u_{1z}(x,y,z,t)}{\partial z^2}\right]-K\beta\frac{\partial T(x,y,z,t)}{\partial x}$$

$$\rho\frac{\partial^2 u_{2y}(x,y,z,t)}{\partial t^2}=\frac{E}{2(1+\gamma)}\left[\frac{\partial^2 u_{1y}(x,y,z,t)}{\partial x^2}+\frac{\partial^2 u_{1x}(x,y,z,t)}{\partial x\partial y}\right]+\left[K+\frac{5E}{12(1+\gamma)}\right]\times$$

$$\frac{\partial^2 u_{1y}(x,y,z,t)}{\partial y^2}+\frac{\partial}{\partial z}\left\{\frac{E}{2(1+\gamma)}\left[\frac{\partial u_{1y}(x,y,z,t)}{\partial z}+\frac{\partial u_{1z}(x,y,z,t)}{\partial y}\right]\right\}+\left[K-\frac{E}{6(1+\gamma)}\right]\times$$

$$\times\frac{\partial^2 u_{1y}(x,y,z,t)}{\partial y\partial z}+K\frac{\partial^2 u_{1y}(x,y,z,t)}{\partial x\partial y}-K\beta\frac{\partial T(x,y,z,t)}{\partial y},$$

$$\rho\frac{\partial^2 u_{2z}(x,y,z,t)}{\partial t^2}=\frac{E}{2(1+\gamma)}\left[\frac{\partial^2 u_{1z}(x,y,z,t)}{\partial x^2}+\frac{\partial^2 u_{1z}(x,y,z,t)}{\partial y^2}+\frac{\partial^2 u_{1x}(x,y,z,t)}{\partial x\partial z}+\right.$$

$$\left.\frac{\partial^2 u_{1y}(x,y,z,t)}{\partial y\partial z}\right]+\frac{\partial}{\partial z}\left\{K(z)\left[\frac{\partial u_{1x}(x,y,z,t)}{\partial x}+\frac{\partial u_{1y}(x,y,z,t)}{\partial y}+\frac{\partial u_{1x}(x,y,z,t)}{\partial z}\right]\right\}+$$

$$\frac{\partial}{\partial z}\left\{\frac{E}{6(1+\gamma)}\left[6\frac{\partial u_{1z}(x,y,z,t)}{\partial z}-\frac{\partial u_{1x}(x,y,z,t)}{\partial x}-\frac{\partial u_{1y}(x,y,z,t)}{\partial y}-\frac{\partial u_{1z}(x,y,z,t)}{\partial z}\right]\right\}-$$

$$K\beta\frac{\partial T(x,y,z,t)}{\partial z}.$$

Integration of the left and right sides of the above equations at time t leads to the following results.

$$u_{2x}(x,y,z,t)=\int_0^t\int_0^\vartheta \frac{1}{\rho}\left[K+\frac{5E}{6(1+\gamma)}\right]\frac{\partial^2 u_{1x}(x,y,z,\tau)}{\partial x^2}d\,\tau\, d\,\vartheta+\int_0^t\int_0^\vartheta \frac{1}{\rho}\left[K-\frac{E}{3(1+\gamma)}\right]\times$$

$$\times\frac{\partial^2 u_{1y}(x,y,z,\tau)}{\partial x\partial y}d\,\tau\, d\,\vartheta+\frac{1}{2}\int_0^t\int_0^\vartheta \frac{E}{1+\gamma}\frac{\partial^2 u_{1y}(x,y,z,\tau)}{\partial y^2}\,d\,\tau\, d\,\vartheta+\frac{1}{2}\int_0^t\int_0^\vartheta \frac{\partial^2 u_{1z}(x,y,z,\tau)}{\partial z^2}\times$$

$$\times\frac{E\,d\,\tau}{1+\gamma}d\,\vartheta+\int_0^t\int_0^\vartheta \frac{1}{\rho}\frac{\partial^2 u_{1z}(x,y,z,\tau)}{\partial x\partial z}\left[K+\frac{E}{3(1+\sigma)}\right]d\,\tau\, d\,\vartheta-\int_0^t\int_0^\vartheta \frac{K\,\beta}{\rho}\frac{\partial\, T(x,y,z,\tau)}{\partial x}\,d\,\tau\, d\,\vartheta-$$

$$-\int_0^\infty\int_0^\vartheta \frac{1}{\rho}\left[K+\frac{5E}{6(1+\sigma)}\right]\frac{\partial^2 u_{1x}(x,y,z,\tau)}{\partial x^2}d\,\tau\, d\,\vartheta-\int_0^\infty\int_0^\vartheta \frac{1}{\rho}\left[K-\frac{E}{3(1+\sigma)}\right]\frac{\partial^2 u_{1y}(x,y,z,\tau)}{\partial x\partial y}d\,\tau\, d\,\vartheta-$$

$$-\frac{1}{2}\int_0^\infty\int_0^\vartheta \frac{E}{\rho(1+\gamma)}\frac{\partial^2 u_{1y}(x,y,z,\tau)}{\partial y^2}d\,\tau\, d\,\vartheta-\int_0^\infty\int_0^\vartheta \frac{1}{\rho(z)}\left[K+\frac{E}{3(1+\gamma)}\right]\frac{\partial^2 u_{1z}(x,y,z,\tau)}{\partial x\partial z}d\,\tau\, d\,\vartheta-$$

$$-\frac{1}{2}\int_0^\infty\int_0^\vartheta \frac{\partial^2 u_{1z}(x,y,z,\tau)}{\partial z^2}\frac{E\,d\,\tau\, d\,\vartheta}{\rho(1+\gamma)}+\int_0^\infty\int_0^\vartheta \frac{K\,\beta}{\rho}\frac{\partial T(x,y,z,\tau)}{\partial x}\,d\,\tau\, d\,\vartheta+u_{0x}$$

$$u_{2y}(x,y,z,t)=\frac{1}{2}\Bigg[\int_0^t\int_0^\vartheta \frac{E}{\rho(1+\gamma)}\frac{\partial^2 u_{1x}(x,y,z,\tau)}{\partial x^2}d\,\tau\, d\,\vartheta+\int_0^t\int_0^\vartheta \frac{E}{\rho(1+\gamma)}\frac{\partial^2 u_{1x}(x,y,z,\tau)}{\partial x\partial y}d\,\tau\, d\,\vartheta\Bigg]+$$

$$+\int_0^t\int_0^\vartheta \frac{1}{\rho}\left[\frac{5E}{12(1+\gamma)}+K\right]\frac{\partial^2 u_{1x}(x,y,z,\tau)}{\partial y^2}d\,\tau\, d\,\vartheta+\int_0^t\int_0^\vartheta \frac{K}{\rho}\frac{\partial^2 u_{1y}(x,y,z,\tau)}{\partial x\partial y}d\,\tau\, d\,\vartheta+$$

$$+\frac{1}{2}\frac{\partial}{\partial z}\Bigg[\int_0^t\int_0^\vartheta \frac{E}{\rho(1+\gamma)}\frac{\partial u_{1y}(x,y,z,\tau)}{\partial z}\,d\,\tau\, d\,\vartheta+\int_0^t\int_0^\vartheta \frac{E}{\rho(1+\gamma)}\frac{\partial u_{1z}(x,y,z,\tau)}{\partial y}d\,\tau\, d\,\vartheta\Bigg]-$$

$$-\int_0^t\int_0^\vartheta \frac{K\,\beta}{\rho}T\,(x,y,z,\tau)\,d\,\tau\, d\,\vartheta+\int_0^t\int_0^\vartheta \frac{1}{\rho}\left[K-\frac{E}{6\,(1+\gamma)}\right]\frac{\partial^2 u_{1y}(x,y,z,\tau)}{\partial y\partial z}d\,\tau\, d\,\vartheta-$$

$$-\frac{1}{2}\Bigg[\int_0^\infty\int_0^\vartheta \frac{E}{\rho(1+\gamma)}\frac{\partial^2\, u_{1x}(x,y,z,\tau)}{\partial x^2}d\,\tau\, d\,\vartheta+\int_0^\infty\int_0^\vartheta \frac{E}{\rho(1+\gamma)}\frac{\partial^2 u_{1x}(x,y,z,\tau)}{\partial x\partial y}d\,\tau\, d\,\vartheta\Bigg]-$$

$$-\int_0^\infty\int_0^\vartheta \frac{K\,\beta}{\rho}T\,(x,y,z,\tau)\,d\,\tau\, d\,\vartheta-\int_0^\infty\int_0^\vartheta \frac{K}{\rho}\frac{\partial^2 u_{1y}(x,y,z,\tau)}{\partial x\partial y}d\,\tau\, d\,\vartheta-\int_0^\infty\int_0^\vartheta\left[K+\frac{5\,E}{12(1+\gamma)}\right]\times$$

$$\times\frac{1}{\rho}\,\frac{\partial^2 u_{1x}(x,y,z,\tau)}{\partial y^2}d\,\tau\, d\,\vartheta-\frac{1}{2}\frac{\partial}{\partial z}\Bigg[\int_0^\infty\int_0^\vartheta \frac{E}{\rho(1+\gamma)}\frac{\partial u_{1y}(x,y,z,\tau)}{\partial z}\,d\,\tau\, d\,\vartheta+\int_0^\infty\int_0^\vartheta \frac{E}{\rho(1+\gamma)}\times$$

$$\times\frac{\partial u_{1z}(x,y,z,\tau)}{\partial y}d\,\tau\, d\,\vartheta\Bigg]-\int_0^\infty\int_0^\vartheta \frac{1}{\rho}\left[K-\frac{E}{6(1+\gamma)}\right]\frac{\partial^2 u_{1y}(x,y,z,\tau)}{\partial y\partial z}\,d\,\tau\, d\,\vartheta+u_{0y}$$

$$u_z(x,y,z,t)=\frac{1}{2}\Bigg[\int_0^\infty\int_0^\vartheta \frac{\partial^2 u_{1z}(x,y,z,\tau)}{\partial x^2}\frac{E\,d\,\tau\, d\,\vartheta}{\rho(1+\gamma)}+\int_0^\infty\int_0^\vartheta \frac{E}{\rho(1+\gamma)}\frac{\partial^2 u_{1z}(x,y,z,\tau)}{\partial y^2}\,d\,\tau\, d\,\vartheta+$$

$$+\int_0^\infty\int_0^\vartheta \frac{E}{\rho(1+\gamma)}\frac{\partial^2 u_{1x}(x,y,z,\tau)}{\partial x\partial z}\,d\,\tau\, d\,\vartheta+\int_0^\infty\int_0^\vartheta \frac{E}{\rho(1+\gamma)}\frac{\partial^2 u_{1y}(x,y,z,\tau)}{\partial y\partial z}\,d\,\tau\, d\,\vartheta\Bigg]+$$

$$+\frac{\partial}{\partial z}\left[\int\limits_0^\infty\int\limits_0^\vartheta \frac{K}{\rho}\frac{\partial u_{1x}(x,y,z,\tau)}{\partial x}d\tau\, d\vartheta+\int\limits_0^\infty\int\limits_0^\vartheta \frac{K}{\rho}\frac{\partial\, u_{1x}(x,y,z,\tau)}{\partial y}\,d\tau\, d\vartheta+\int\limits_0^\infty\int\limits_0^\vartheta \frac{K}{\rho}\times\right.$$

$$\left.\times\frac{\partial u_{1x}(x,y,z,\tau)}{\partial z}\,d\tau\, d\vartheta\right]+\frac{\partial}{\partial z}\left[\int\limits_0^\infty\int\limits_0^\vartheta \frac{\partial u_{1z}(x,y,z,\tau)}{\partial z}\frac{E\, d\tau\, d\vartheta}{\rho(1+\gamma)}-\frac{1}{6}\int\limits_0^\infty\int\limits_0^\vartheta \frac{\partial u_{1x}(x,y,z,\tau)}{\partial x}\times\right.$$

$$\left.\times\frac{E\, d\tau\, d\vartheta}{\rho(1+\gamma)}-\frac{1}{6}\int\limits_0^\infty\int\limits_0^\vartheta \frac{\partial u_{1y}(x,y,z,\tau)}{\partial y}\frac{E\, d\tau\, d\vartheta}{\rho(1+\gamma)}-\frac{1}{6}\int\limits_0^\infty\int\limits_0^\vartheta \frac{\partial u_{1z}(x,y,z,\tau)}{\partial z}\frac{E\, d\tau\, d\vartheta}{\rho(1+\gamma)}\right]-$$

$$-\int\limits_0^\infty\int\limits_0^\vartheta \frac{K\beta}{\rho}\frac{\partial T(x,y,z,\tau)}{\partial z}d\tau\, d\vartheta+u_{0z}.$$

In the framework of this paper, we calculate components of the displacement vector and the spatio-temporal distribution of temperature by using the second-order approximation in the framework of the method of averaging of function corrections. The approximation is usually good enough to make qualitative analysis and to obtain some quantitative results. All obtained results have been checked by comparison with the results of numerical simulations.

3 | Discussion

In this section, we analyzed the change in stress in metals during thermal treatment. Based on the analysis, we estimate residual stress after finishing the thermal treatment. *Fig. 1* shows typical dependences of components of the displacement vector on the coordinate. In this figure, as an example, we consider the dependencies of component u_z on coordinate z. An increase in the number of curves and a decrease in the coordinate correspond to an increase in the defects of metals (dislocations, pores). The defects became a reason for residual stress in metals.

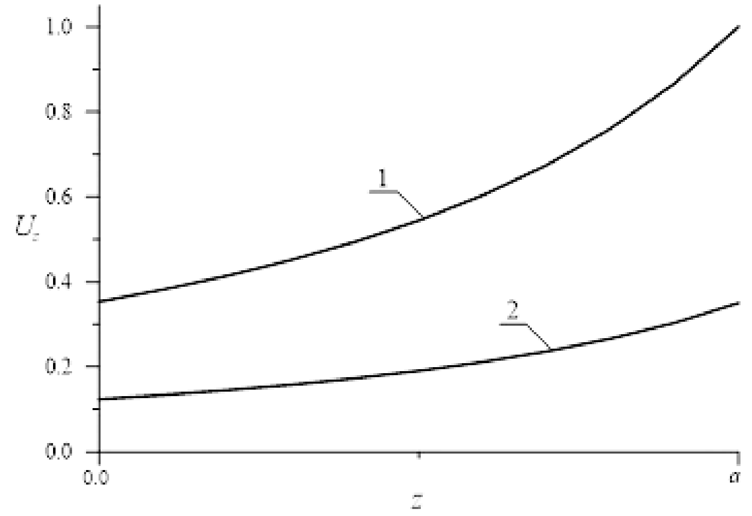

Fig. 1. Normalized dependences of component u_z of displacement vector on coordinate z. An increase in the number of curves and a decrease in the coordinate correspond to an increase in the defectiveness of metals.

4 | Conclusion

In this paper, we introduce an approach for the estimation of stresses in metals during heat treatment and residual stresses in these materials after the treatment. In the framework of the considered approach, we introduce a model of stress in metals during heat treatment as well as an analytical approach for its analysis.

Authors' Contributions

All aspects of the research and manuscript preparation were carried out by the author. The author has read and approved the final version of the manuscript.

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All data supporting the reported findings in this research paper are provided within the manuscript.

Conflict of Interest

The author declares that they do not have any conflict of interest.

Consent for Publication

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Ethics Approval and Consent to Participate

This article does not contain any studies with human participants performed by the author.

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