Karshi Multidisciplinary International Scientific Journal

Vol. 1(1) (2024), 80–87

ON THE BEHAVIOR OF SHORT G. WEYL TRIGONOMETRIC SUMS ON MAJOR ARCS

Alisher Orzu Rakhimzoda

Communicated by Irfan Nurhidayat

Abstract. This paper studies the behavior of G. Weyl's short trigonometric sums on large arcs.

Keywords and phrases: Short trigonometric sum, nontrivial estimate, large arcs, asymptotic formula

MSC 2010 Classifications: Primary 11L07; Secondary 11L03.

1 Introduction

The study of exponential sums occupies a central position in analytic number theory due to their deep connections with problems in Diophantine approximation, the distribution of prime numbers, and Waring–type additive questions. Among these, G. Weyl's trigonometric sums play a particularly important role, providing powerful analytic tools for understanding arithmetic sequences through oscillatory behavior. Classical results by Hua Loo-Keng and R. Vaughan established fundamental estimates for complete and incomplete Weyl sums, which later became indispensable in deriving asymptotic formulas for higher–degree Diophantine equations and for improving bounds in Waring's problem. In many applications, it becomes essential to study short Weyl sums, where summation is restricted to a small interval instead of beginning at the origin. These short sums capture more delicate local behavior and appear naturally in problems with "almost equal" summands, in the analysis of shifted polynomial sequences, and in the investigation of exponential sums on major and minor arcs. Earlier results addressed only small exponents or special cases; however, recent progress has extended these methods to arbitrary fixed degree n. The aim of this article is to refine and simplify the existing analysis of short G. Weyl trigonometric sums on major arcs. Building on the approach of Vaughan and subsequent developments by Rakhmonov and collaborators, we obtain new estimates for the sums $T(\alpha; x, y)$ and establish an asymptotic formula under natural Diophantine conditions. Our results generalize previous work and contribute to a clearer understanding of the structure and behavior of short Weyl sums in analytic number theory.

2 Materials and methodology

R. Vaughan [1], studying G. Weyl sums of the form

$$T(\alpha, x) = \sum_{m \le x} e(\alpha m^n), \quad \alpha = \frac{a}{q} + \lambda, \quad q \le \tau, \quad (a, q) = 1, \quad |\lambda| \le \frac{1}{q\tau},$$

used on major arcs the estimate

$$S_b(a, q) = \sum_{k=1}^{q} e\left(\frac{ak^n + bk}{q}\right) \ll q^{\frac{1}{2}+\varepsilon}(b, q), \tag{2.1}$$

due to Hua Loo-Keng [2], and by means of van der Corput's method proved:

$$T(\alpha, x) = \frac{S(a, q)}{q} \int_0^x e(\lambda t^n)\, dt + O\left(q^{\frac{1}{2}+\varepsilon}\left(1 + x^n|\lambda|\right)^{\frac{1}{2}}\right). \quad S(a, q) = S_0(a, q),$$

Under the condition that α is very well approximated by a rational number with denominator q, that is, when

$$|\lambda| \leq \frac{1}{2nqx^{n-1}},$$

he also proved:

$$T(\alpha, x) = \frac{x\, S(a,q)}{q} \int_0^1 e\left(\lambda t^n\right) dt + O\left(q^{\frac{1}{2}+\varepsilon}\right).$$

He used these estimates to derive an asymptotic formula in Waring's problem for eight cubes [3].

Short G. Weyl trigonometric sums of the form

$$T(\alpha; x, y) = \sum_{x-y<m\leq x} e(\alpha m^n), \quad \alpha = \frac{a}{q} + \lambda,\ q \leq \tau, \quad (a,q) = 1,\ |\lambda| \leq \frac{1}{q\tau}, \tag{2.2}$$

obtained from $T(\alpha, x)$ by replacing the condition $m \leq x$ with $x - y < m \leq x$, were investigated on major arcs for $n = 2, 3, 4$ in [4, 5] and applied in deriving asymptotic formulas with almost equal summands in Waring's problem (for cubes and fourth powers) and in Estermann's cubic problem in [6]. Later, for arbitrary fixed n the sum $T(\alpha; x, y)$ was studied in [7, 8]. The main result of this work is a simplification of the proof and refinement of the main theorem from [7, 8].

3 Main part. Statement of the scientific problem.

Theorem 3.1. *Let $\tau \geq 2n(n-1)x^{n-2}y$ and $\lambda \geq 0$. Then if $\{n\lambda x^{n-1}\} \leq \frac{1}{2q}$, the formula*

$$T(\alpha, x, y) = \frac{S(a,q)}{q} T(\lambda; x, y) + O(q^{\frac{1}{2}+\varepsilon})$$

holds, and if $\{n\lambda x^{n-1}\} > \frac{1}{2q}$, the estimate

$$|T(\alpha, x, y)| \ll q^{1-\frac{1}{n}} \ln q + \min_{2\leq k\leq n} (yq^{-\frac{1}{n}}, \lambda^{-\frac{1}{k}} x^{1-\frac{n}{k}} q^{-\frac{1}{n}})$$

holds.

Corollary 3.2. *Let $\tau \geq 2n(n-1)x^{n-2}y$, $|\lambda| \leq \frac{1}{2nqx^{n-1}}$. Then the relation*

$$T(\alpha, x, y) = \frac{y}{q} S(a,q)\gamma(\lambda; x, y) + O(q^{\frac{1}{2}+\varepsilon}),$$

$$\gamma(\lambda; x, y) = \int_{-0.5}^{0.5} e\left(\lambda\left(x - \frac{y}{2} + yt\right)^n\right) dt.$$

Corollary 3.3. *Let $\tau \geq 2n(n-1)x^{n-2}y$, $\frac{1}{2nqx^{n-1}} < |\lambda| \leq \frac{1}{q\tau}$. Then we have the estimate*

$$T(\alpha, x, y) \ll q^{1-\frac{1}{n}} \ln q + \min_{2\leq k\leq n} \left(yq^{-\frac{1}{n}}, x^{1-\frac{1}{k}} q^{\frac{1}{k}-\frac{1}{n}}\right).$$

Corollaries 3.2 and 3.3 generalize R. Vaughan's results [1] for short G. Weyl trigonometric sums $T(\alpha; x, y)$ of the form (2.2).

The proof of Theorem 3.1 is carried out by estimating special van der Corput trigonometric sums, using Poisson summation, estimates of trigonometric integrals via the size of derivatives, and the estimate of complete rational sums (2.1) due to Hua Loo-Keng [2].

Proof of Theorem 3.1. Using the orthogonality property of complete linear rational trigonometric sums, we find

$$T(\alpha;x,y)=\sum_{x-y<m\le x}e\left(\frac{ak^n}{q}+\lambda m^n\right)\sum_{\substack{k=1\\ k\equiv m(mod q)}}^{q}1=$$
$$=\sum_{k=1}^{q}e\left(\frac{ak^n}{q}\right)\sum_{\substack{x-y<m\le x\\ m\equiv k(mod q)}}e(\lambda m^n)=$$
$$=\sum_{k=1}^{q}e\left(\frac{ak^n}{q}\right)\sum_{x-y<m\le x}e(\lambda m^n)\frac{1}{q}\sum_{b=1}^{q}e\left(\frac{b(k-m)}{q}\right)=$$
$$=\frac{1}{q}\sum_{b=1}^{q}T_b(\lambda;x,y)S_b(a,q), \tag{3.1}$$

where

$$T_b(\lambda;x,y)=\sum_{x-y<m\le x}e\left(\lambda m^n-\frac{bm}{q}\right),\quad T(\lambda;x,y)=T_0(\lambda;x,y),$$
$$S_b(a,q)=\sum_{k=1}^{q}e\left(\frac{ak^n+bk}{q}\right),\quad S(a,q)=S_0(a,q).$$

Let $R(\alpha;x,y)$ denote the part of $T(\alpha;x,y)$ defined by (3.1), with the term $b=q$ omitted:

$$R(\alpha;x,y)=\frac{1}{q}\sum_{b=1}^{q-1}T_b(\lambda;x,y)S_b(a,q). \tag{3.2}$$

Noting that $n\lambda x^{n-1}-\{n\lambda x^{n-1}\}$ is an integer, represent

$$T_b(\lambda;x,y)=\sum_{x-y<m\le x}e\left(f(m,b)\right),$$
$$f(u,b)=\lambda u^n-(n\lambda x^{n-1}-\{n\lambda x^{n-1}\})u-\frac{bu}{q}.$$

We compute the first and second derivatives:

$$f'(u,b)=n\lambda(u^{n-1}-x^{n-1})+\{n\lambda x^{n-1}\}-\frac{b}{q},$$
$$f''(u,b)=n(n-1)\lambda u^{n-2}\ge 0.$$

Hence $f'(u,b)$ is nondecreasing on $u\in(x-y,x]$, therefore

$$f'(x-y,b)<f'(u,b)\le f'(x,b). \tag{3.3}$$

Estimating $f'(x,b)$ from above:

$$f'(x,b)=\{n\lambda x^{n-1}\}-\frac{b}{q}<1-\frac{b}{q}, \tag{3.4}$$

To obtain a lower bound for $f'(x-y,b)$, we use the representation

$$f'(x-y,b)=-n\lambda\left(x^{n-1}-(x-y)^{n-1}\right)+\{n\lambda x^{n-1}\}-\frac{b}{q}=$$
$$=n\lambda\sum_{k=1}^{n-1}(-1)^kC_{n-1}^kx^{n-1-k}y^k+\{n\lambda x^{n-1}\}-\frac{b}{q}=$$
$$=-n(n-1)\lambda x^{n-2}y+n\lambda\sum_{k=2}^{n-1}(-1)^kC_{n-1}^kx^{n-1-k}y^k+\{n\lambda x^{n-1}\}-\frac{b}{q}.$$

Using the monotonicity of $f'(u,b)$, the condition

$$\tau \geq 2n(n-1)x^{n-2}y,$$

and the inequality

$$W = \sum_{k=2}^{n-1}(-1)^k C_{n-1}^k x^{n-1-k} y^k \geq 0, \qquad n \geq 3, \qquad 3x \geq (n-3)y,$$

we obtain

$$f'(u,b) \leq f'(x,b) = \{n\lambda x^{n-1}\} - \frac{b}{q} < 1,$$

$$f'(u,b) \geq f'(x-y,b) = -n(n-1)\lambda x^{n-2}y + n\lambda W + \{n\lambda x^{n-1}\} - \frac{b}{q} \geq$$

$$\geq -n(n-1)\lambda x^{n-2}y - \frac{b}{q} \geq -\frac{n(n-1)x^{n-2}y}{q\tau} - \frac{b}{q} \geq -1 + \frac{1}{2q}.$$

Therefore, applying the Poisson summation formula to the sum $T_b(\lambda; x, y)$ with $\alpha = -1$, $\beta = 1$, $\varepsilon = 0.5$, we obtain

$$T_b(\lambda; x, y) = I(-1,b) + I(0,b) + I(1,b) + O(1), \tag{3.5}$$

$$I(h,b) = \int_{x-y}^{x} e(f_h(u,b))\, du, \qquad f_h(u,b) = f(u,b) - hu,$$

The function

$$f'_h(u,b) = n\lambda(u^{n-1} - x^{n-1}) + \{n\lambda x^{n-1}\} - \frac{b}{q} - h$$

is nondecreasing on the interval $u \in [x-y, x]$. Therefore,

$$f'_h(x-y,b) \leq f'_h(u,b) \leq f'_h(x,b),$$

which may be written in the form

$$\{n\lambda x^{n-1}\} - \frac{b}{q} - h - \eta < f'_h(u,b) \leq \{n\lambda x^{n-1}\} - \frac{b}{q} - h, \tag{3.6}$$

$$\eta = n(n-1)\lambda x^{n-2}y - n\lambda W \leq n(n-1)\lambda x^{n-2}y \leq \frac{n(n-1)x^{n-2}y}{q\tau} \leq \frac{1}{2q}.$$

Substituting (3.5) into (3.1) and (3.2), we obtain

$$T(\alpha; x, y) = T_{-1} + T_0 + T_1 + O\left(\frac{1}{q}\sum_{b=0}^{q-1}|S_b(a,q)|\right), \tag{3.7}$$

$$R(\alpha; x, y) = R_{-1} + R_0 + R_1 + O\left(\frac{1}{q}\sum_{b=1}^{q-1}|S_b(a,q)|\right), \tag{3.8}$$

$$T_h = \frac{1}{q}\sum_{b=0}^{q-1} I(h,b)\, S_b(a,q),$$

$$R_h = \frac{1}{q}\sum_{b=1}^{q-1} I(h,b)\, S_b(a,q).$$

Using estimate (2.1), we bound the remainder term:

$$\frac{1}{q}\sum_{b=1}^{q-1}|S_b(a,q)| \ll q^{-\frac{1}{2}+\varepsilon}\sum_{b=1}^{q-1}(b,q) = q^{-\frac{1}{2}+\varepsilon}\sum_{\delta|q}\delta \sum_{\substack{1\leq b\leq q-1 \\ (b,q)=\delta}} 1 \leq q^{\frac{1}{2}+\varepsilon}\tau(q).$$

We now estimate each of the sums T_h and R_h separately.
Estimate of T_1 and R_1. Setting $h = 1$ in (3.6), we obtain

$$f_1'(u,b) \le \{n\lambda x^{n-1}\} - \frac{b}{q} - 1 \le -\frac{b}{q} < 0.$$

Estimating the integral via the magnitude of the first derivative, we obtain

$$|I(1,b)| = \left|\int_{x-y}^{x} e(f_1(u,b))\,du\right| \ll \frac{q}{b}.$$

From this and from (2.1), we obtain

$$R_1 = \frac{1}{q}\sum_{b=1}^{q-1} I(1,b)S_b(a,q) \ll \sum_{b=1}^{q-1}\frac{|S_b(a,q)|}{b} \ll q^{\frac{1}{2}+\varepsilon}\sum_{b=1}^{q-1}\frac{(b,q)}{b} \ll q^{\frac{1}{2}+2\varepsilon}.$$

In the case $b = 0$, using the inequality

$$f_1^{(k)}(u,q) \ge n(n-1)\cdots(n-k+1)\lambda(x-y)^{n-k} \gg \lambda x^{n-k}, \qquad k = 2,3,\ldots,n,$$

and estimating the integral $I(1,0)$ via the magnitude of the kth derivative, we find

$$|I(1,0)| \ll \min_{2\le k\le n}\left(y,\ \lambda^{-\frac{1}{k}}x^{1-\frac{n}{k}}\right).$$

From this, and using the estimate $|S(a,q)| \ll q^{1-\frac{1}{n}}$, taking into account the estimate for R_1, we obtain

$$T_1 \le |R_1| + \frac{|I(1,0)|\,|S(a,q)|}{q} \ll q^{\frac{1}{2}+2\varepsilon} + \min_{2\le k\le n}\left(yq^{-\frac{1}{n}},\ \lambda^{-\frac{1}{k}}x^{1-\frac{n}{k}}q^{-\frac{1}{n}}\right).$$

Estimate of T_{-1} and R_{-1}. Setting $h = -1$ in (3.6), we obtain

$$f_{-1}'(u,b) > \{n\lambda x^{n-1}\} + \frac{q-b}{q} - \eta \ge \frac{q-b}{q}.$$

The integral $I(-1,b)$ is also estimated using the magnitude of the first derivative. We have

$$|I(-1,b)| = \left|\int_{x-y}^{x} e(f_{-1}(u,b))\,du\right| \ll \frac{q}{q-b}.$$

Proceeding analogously to the case of estimating R_1, we obtain

$$R_{-1} = \sum_{b=1}^{q-1}\frac{I(-1,b)\,S_b(a,q)}{q} \ll \sum_{b=1}^{q-1}\frac{|S_b(a,q)|}{q-b} \ll q^{\frac{1}{2}+\varepsilon}\sum_{b=1}^{q-1}\frac{(b,q)}{b} \ll q^{\frac{1}{2}+2\varepsilon}.$$

$$T_{-1} \le |R_{-1}| + \frac{|I(-1,0)|\,|S(a,q)|}{q} \ll q^{\frac{1}{2}+2\varepsilon} + \frac{|S(a,q)|}{q} \ll q^{\frac{1}{2}+2\varepsilon}.$$

Estimate of R_0. If $\{n\lambda x^{n-1}\} \le \frac{1}{2q}$, then setting $h = 0$ in (3.6), we obtain

$$f_0'(u,b) \le \{n\lambda x^{n-1}\} - \frac{b}{q} \le \frac{1-2b}{2q} \le -\frac{b}{2q} < 0.$$

Estimating the integral $I(0,b)$ again using the magnitude of the first derivative, we find

$$|I(0,b)| = \left|\int_{x-y}^{x} e(f_0(u,b))\,du\right| \ll \frac{q}{b}.$$

Proceeding similarly to the estimate for R_1, we obtain

$$R_0 = \sum_{b=1}^{q-1}\frac{I(0,b)\,S_b(a,q)}{q} \ll \sum_{b=1}^{q-1}\frac{|S_b(a,q)|}{b} \ll q^{\frac{1}{2}+\varepsilon}\sum_{b=1}^{q-1}\frac{(b,q)}{b} \ll q^{\frac{1}{2}+2\varepsilon}.$$

From this, together with the estimates for R_1 and R_{-1} and using (3.8), we obtain the first assertion of the theorem.

Estimate of T_0. When $\{n\lambda x^{n-1}\} \geq \frac{1}{2q}$, we define a natural number r by the relation

$$\frac{r}{2q} \leq \{n\lambda x^{n-1}\} < \frac{r+1}{2q}, \qquad 1 \leq r \leq 2q-1.$$

From this, using inequality (3.6) with $h = 0$ and the condition $\eta \leq \frac{1}{2q}$, we obtain

$$f_0'(u,b) > \{n\lambda x^{n-1}\} - \frac{b}{q} - \eta \geq \frac{r-2b-1}{2q}, \tag{3.9}$$

$$f_0'(u,b) \leq \{n\lambda x^{n-1}\} - \frac{b}{q} < \frac{r-2b+1}{2q}. \tag{3.10}$$

Let $r = 2r_1$ be even ($1 \leq r_1 \leq q-1$). We split the summation interval $0 \leq b \leq q-1$ in the sum T_0 into the following three sets:

$$0 \leq b \leq r_1 - 1, \qquad b = r_1, \qquad r_1 + 1 \leq b \leq q-1,$$

in the first of which the right-hand side of inequality (3.9) is positive, and in the third the right-hand side of (3.10) is negative, that is,

$$f_0'(u,b) > \frac{2r_1 - 2b - 1}{2q} \geq \frac{r_1 - b}{2q}, \qquad 0 \leq b \leq r_1 - 1,$$

$$f_0'(u,b) < \frac{2r_1 - 2b + 1}{2q} \leq \frac{r_1 - b}{2q}, \qquad r_1 + 1 \leq b \leq q-1.$$

Using these inequalities and estimating the integral $I(0,b)$ via the size of the first derivative, we obtain

$$I(0,b) = \int_{x-y}^{x} e(f_0(u,b))\,du \ll \frac{q}{|r_1 - b|}, \qquad b \neq r_1.$$

In the case $b = r_1$, evaluating as in the estimate of the integral $I(1,0)$, we find

$$|I(0,r_1)| \ll \min_{2 \leq k \leq n} \left(y,\ \lambda^{-\frac{1}{k}} x^{1-\frac{n}{k}}\right).$$

Using these estimates together with the bound $|S(a,q)| \ll q^{1-\frac{1}{n}}$, we obtain

$$T_0 = \sum_{b=0}^{q-1} \frac{I(0,b) S_b(a,q)}{q} \ll q^{-\frac{1}{n}} \left(\sum_{\substack{b=0 \\ b \neq r_1}}^{q-1} \frac{q}{|r_1 - b|} + \min_{2 \leq k \leq n} \left(y,\ \lambda^{-\frac{1}{k}} x^{1-\frac{n}{k}}\right) \right)$$

$$\ll q^{1-\frac{1}{n}} \ln q + \min_{2 \leq k \leq n} \left(y q^{-\frac{1}{n}},\ \lambda^{-\frac{1}{k}} x^{1-\frac{n}{k}} q^{-\frac{1}{n}}\right).$$

Let now $r = 2r_1 + 1$ be odd ($0 \leq r_1 \leq q-1$). We split the summation interval $0 \leq b \leq q-1$ in the sum R_0 into the following three sets:

$$0 \leq b \leq r_1 - 1, \qquad b = r_1,\ r_1 + 1, \qquad r_1 + 2 \leq b \leq q-1,$$

in the first of which the right-hand side of inequality (3.9) is positive, and in the third the right-hand side of (3.10) is negative, that is,

$$f_0'(u,b) > \frac{2r_1 + 1 - 2b - 1}{2q} = \frac{r_1 - b}{q}, \qquad 0 \leq b \leq r_1 - 1,$$

$$f_0'(u,b) < \frac{2r_1 + 1 - 2b + 1}{2q} \leq \frac{r_1 - b}{2q}, \qquad r_1 + 2 \leq b \leq q-1.$$

Consequently,

$$I(0,b)=\int_{x-y}^{x} e(f_0(u,b))\,du \ll \frac{q}{|r_1-b|}, \qquad b\neq r_1-1,\ r_1.$$

In the case $b=r_1-1,\ r_1$, proceeding as in the previous estimate of $I(0,r_1)$, we obtain

$$|I(0,b)| \ll \min_{2\le k\le n}\left(y,\ \lambda^{-\frac{1}{k}}x^{1-\frac{n}{k}}\right), \qquad b=r_1,\ r_1+1.$$

From these estimates for $I(0,b)$ we obtain

$$\begin{aligned} T_0 &\le \frac{1}{q}\sum_{b=0}^{q-1}|I(0,b)|\,|S_b(a,q)| \\ &\ll q^{-\frac{1}{n}}\left(\sum_{\substack{b=0\\ b\neq r_1,\,r_1+1}}^{q-1}\frac{q}{|r_1-b|}+\min_{2\le k\le n}\left(y,\ \lambda^{-\frac{1}{k}}x^{1-\frac{n}{k}}\right)\right) \\ &\ll q^{1-\frac{1}{n}}\ln q+\min_{2\le k\le n}\left(yq^{-\frac{1}{n}},\ \lambda^{-\frac{1}{k}}x^{1-\frac{n}{k}}q^{-\frac{1}{n}}\right). \end{aligned}$$

Substituting the estimates for T_1, T_{-1}, and T_0 into (3.7), we obtain the second assertion of the theorem.

REMARK. The case $\lambda<0$ reduces to the case $\lambda\ge 0$ if we rewrite formula (3.1) in the form

$$\overline{T(\alpha;x,y)}=\frac{1}{q}\sum_{b=0}^{q-1}T_{q-b}(-\lambda;x,y)\,S_{q-b}(q-a,q)=\frac{1}{q}\sum_{b=0}^{q-1}T_b(-\lambda;x,y)\,S_b(q-a,q).$$

4 Conclusion

In this work, we investigated the behavior of short G. Weyl trigonometric sums on major arcs and refined earlier results obtained by Vaughan and subsequent researchers. By applying a combination of analytical tools-including Poisson summation, derivative-based estimates for oscillatory integrals, and Hua Loo-Keng's classical bounds for complete rational sums—we achieved a clearer and more streamlined derivation of the principal estimates governing these short sums. The approach emphasizes the advantages of analyzing the structure of derivatives and exploiting the monotonicity properties inherent in the phase functions associated with the sums. The main theorem provides two distinct outcomes depending on the approximation properties of the frequency parameter. In the region where the approximation is sufficiently strong, a precise relation between the short Weyl sum and its continuous analogue emerges, allowing the sum to be effectively expressed in terms of complete rational sums. In the complementary region, we obtained nontrivial bounds that extend and generalize Vaughan's earlier estimates. These results not only deepen the understanding of the analytic behavior of short Weyl sums but also strengthen their applicability to classical additive problems, particularly those related to Waring-type questions. Overall, the methods developed here demonstrate that a careful combination of classical analytic number theory techniques can yield significant simplifications and improvements to previously established results. This contributes to a broader framework for approaching multidimensional exponential sums and enhances the toolkit available for tackling related problems in modern analytic number theory.

Acknowledgment

The authors express their sincere gratitude to the reviewers and editors for their useful comments.

References

[1] Vaughan, R. C. Some remarks on Weyl sums. *Colloquium on Number Theory, Math. Soc. János Bolyai*, Budapest, 1981.

[2] Hua, Loo-Keng. The Method of Trigonometric Sums and Its Applications in Number Theory. Moscow: Mir, 1964, 190 pp. (In Russian)

[3] Vaughan, R. C. On Waring's problem for cubes. *Journal für die reine und angewandte Mathematik*, 1986, vol. 365, pp. 122–170.

[4] Rakhmonov, Z. Kh. The ternary Estermann problem with almost equal summands. *Mathematical Notes*, 2003, vol. 74, no. 4, pp. 564–572. (In Russian)

[5] Rakhmonov, Z. Kh., and Shokhamolova, J. A. Short quadratic Weyl-type trigonometric sums. *Proceedings of the Academy of Sciences of the Republic of Tajikistan, Series of Physical-Mathematical, Chemical, Geological and Technical Sciences*, 2009, no. 2(135), pp. 7–18. (In Russian)

[6] Rakhmonov, Z. Kh., and Fozilova, D. M. On a ternary problem with almost equal summands. *Reports of the Academy of Sciences of the Republic of Tajikistan*, 2012, vol. 55, no. 6, pp. 433–440. (In Russian)

[7] Rakhmonov, Z. Kh., and Ozodbekova, N. B. An estimate for short Weyl-type trigonometric sums. *Reports of the Academy of Sciences of the Republic of Tajikistan*, 2011, vol. 54, no. 4, pp. 257–264. (In Russian)

[8] Rakhmonov, Z. Kh. Short Weyl trigonometric sums. *Scientific Notes of Oryol University. Series: Natural, Technical, and Medical Sciences*, 2013, no. 6, part 2, pp. 194–203. (In Russian)

Author information

Alisher Orzu Rakhimzoda, Institute of Mathematics AS Rebublic of Tajikistan, Dushanbe, TAJIKISTAN.
E-mail: rao8787@mail.ru

Received: 01.04.2024
Accepted: 31.05.2024
Published: 30.06.2024