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ON THE ENTROPY OF A HALF SYNCHRONIZING SPACE

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Abstract: To establish the concept of half synchronized entropy, we must first clarify the conditions under which a subshift can be deemed half synchronized. This characterization involves analyzing the block structure inherent in the subshift, focusing on how certain patterns can repeat over time, albeit with some restrictions. Unlike fully synchronized systems where a global periodic structure dominates, half synchronized systems allow for greater complexity while still maintaining a semblance of order. This duality between complexity and order forms the crux of half synchronized entropy. Nearly sofic systems have been known to exhibit rich and intricate behaviors, bridging the gap between symbolic dynamics and more conventional dynamical systems. The exploration of these connections paves the way for further advancements in the study of synchronized systems and their entropy properties.

Keywords and phrases: Synchronized entropy, minimal half synchronizing, half synchronized, entropy.

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1 Introduction

Thomsen in [10] considers a synchronized component of a general subshift and investigates the approximation of entropy from inside of this synchronized component by some certain shift of finite type's. In fact, many results in [10] are based on this result. To be more specified, suppose $W(X)$ is the set of admissible words of X and $W_n(X)$ the set of admissible words of length n or so called n-words. Thomsen proves that $\lim_{k\to\infty} h(X_k) = h_{syn}(X)$ where where m is an arbitrary synchronizing word in $W(X)$ and $h_{syn}(X)$ is half synchronizing entropy also X_k's are shift of finite type approaching X from inside. In section (3), we extend a version of this result to half synchronized systems.

Now suppose $\text{Per}(X)$ be the set of periodic points of X and set $R(X) = \overline{\text{Per}(X)}$. Also suppose $S(X)$ denote the set of synchronizing words for $R(X)$.

Synchronized systems are well known, however, there are not much results for half synchronized systems. In this note, we will consider them and in particular, a natural extension of the concept of h_{syn} will be given for half synchronized systems denoted by h_{hsyn} and we will show that if $h(X) = h_{hsyn}(X)$ then X is almost sofic.

2 Background and definitions

suppose $\mathcal{A}$ be a set of non-empty finite symbols called *alphabet*. The full $\mathcal{A}$-shift denoted by $\mathcal{A}^{\mathbb{Z}}$, is the collection of all bi-infinite sequences of symbols in $\mathcal{A}$. Equip $\mathcal{A}$ with discrete topology and $\mathcal{A}^{\mathbb{Z}}$ with product topology. A *word* over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. It is convenient to include the sequence of no symbols, called the *empty word* which is denoted by ε. If $n \geq 1$, then u^n denotes the concatenation of n copies of u, and put $u^0 = \varepsilon$. The *shift map* σ on the full shift $\mathcal{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose i-th coordinate is $y_i = x_{i+1}$. By our topology, σ is a homeomorphism. Suppose $\mathcal{F}$ be the collection of all forbidden words over $\mathcal{A}$. The complement of $\mathcal{F}$ is the set of *admissible words* or just words in X. For a full shift $\mathcal{A}^{\mathbb{Z}}$, define $X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ not containing any word from $\mathcal{F}$. A *subshift* is a subset X of a full shift $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some collection $\mathcal{F}$ of forbidden words.

Suppose $W_n(X)$ denote the set of all admissible n-words. The *language* of X is the collection $W(X) = \cup_n W_n(X)$. A subshift X is *irreducible* if for every ordered pair of words $u,\ v \in W(X)$

there is a word $w \in W(X)$ so that $uwv \in W(X)$. A subshift X is called a *shift of finite type* if there is a finite set $\mathcal{F}$ of forbidden words such that $X = X_{\mathcal{F}}$.

Suppose G be a graph with edge set $\mathcal{E} = \mathcal{E}(G)$ and the set of vertices $\mathcal{V} = \mathcal{V}(G)$. The *edge shift* X_G is the subshift over the alphabet $\mathcal{A} = \mathcal{E}$ defined by

$$X_G = \{\xi = (\xi_i)_{i\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : t(\xi_i) = i(\xi_{i+1})\}.$$

Each edge e initiates at a vertex denoted by $i(e)$ and terminates at a vertex $t(e)$.

A labeled graph is a pair $\mathcal{G} = (G, \mathcal{L})$, where G is a graph with edge set $\mathcal{E}$, and the labeling $\mathcal{L} : \mathcal{E}(G) \to \mathcal{A}$ assigns to each edge e of G a label $\mathcal{L}(e)$ from the finite alphabet $\mathcal{A}$. For a path $\pi = \pi_0 \dots \pi_k$, $\mathcal{L}(\pi) = \mathcal{L}(\pi_0) \dots \mathcal{L}(\pi_k)$ is the label of π. Suppose $\mathcal{L}_\infty(\xi)$ be the sequence of bi-infinite labels of a bi-infinite path ξ in G and set

$$X_{\mathcal{G}} := \{\mathcal{L}_\infty(\xi) : \ \xi \in X_G\} = \mathcal{L}_\infty(X_G).$$

We say $\mathcal{G}$ is a *presentation* or *cover* for $X = \overline{X_{\mathcal{G}}}$. In particular, X is sofic if, for a finite graph G, have $X = X_{\mathcal{G}}$ [2, Proposition 3.2.10]. Every shift of finite type is sofic [2, Theorem 3.1.5], but the converse is not true [2, Page 67]. A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is *right-resolving* if for each vertex I of G the edges starting at I carry different labels.

Suppose X be a subshift and $w \in W(X)$. The follower set $F(w)$ of w is defined by $F(w) = \{v \in W(X) : wv \in W(X)\}$. Suppose $x \in X$. Then, $x_+ = (x_i)_{i\in Z^+}$ (resp. $x_- = (x_i)_{i\leq 0}$) is called right (resp. left) infinite X-ray. For a left infinite X-ray, say x_-, its follower set is $w_+(x_-) = \{x_+ \in X^+ : x_-x_+ \in X\}$. Consider the collection of all follower sets $w_+(x_-)$ as the set of vertices of a graph. There is an edge from I_1 to I_2 labeled a if and only if there is an X-ray x_- such that x_-a is an X-ray and $I_1 = w_+(x_-)$, $I_2 = w_+(x_-a)$. This labeled graph is called the *Krieger graph* for X.

A word $m \in W(X)$ is *synchronizing* if whenever um and mv are in $W(X)$, we have $umv \in W(X)$. An irreducible subshift X is *synchronized system* if it has a synchronizing word, or equivalently, if and only if it admit a countable generating graph G such that $\mathcal{L}_\infty(X_G)$ is residual in X [3, Theorem 1.1]. A word $m \in W(X)$ is *(left) half synchronizing* if there is a left transitive point $x \in X$ such that $x_{[-|m|+1,\, 0]} = m$ and $w_+(x_{(-\infty,\, 0]}) = w_+(m)$, that we denote by

$$(\!(x,\, m)\!). \tag{2.1}$$

If X is a (left) half synchronized system with (left) half synchronizing m, the irreducible component of the Krieger graph containing the vertex $w_+(m)$ is denoted by X_0^+ and is called the *right Fischer cover* of X. If for some $m \in W(X)$ there is a unique vertex I such that $m \in F_-(m)$, then m is called a *magic* word for the Fischer cover.

Suppose X be a subshift. The *entropy* of X is defined by

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log |W_n(X)|.$$

A subshift X is *almost sofic* if there are sofic shifts $X_n \subseteq X$ such that

$$\lim_{n\to\infty} h(X_n) = h(X).$$

3 Half synchronized components of a subshift

In this section, we will extend half synchronizing entropy to half synchronized components of a subshift with some reservations. First, Suppose us comment that one may verify that without loss of generality $S(X)$ could be taken to be the set of synchronizing words of maximal irreducible components of $R(X)$. For suppose C be such a component and suppose s be synchronizing in C. Then, there is a word u in C that does not appear in any other component. Now a prolongation of s including u is synchronizing in $S(X)$. Suppose X be a subshift and call m a half synchronizing word for X, if m is half synchronizing for an irreducible component of X. (Note that if m were defined for whole system, as in synchronizing case, then X had to have just one component.) Before stating our main proposition, suppose X be a half synchronized system and hence irreducible by definition. Set $H(X)$ to be the set of half synchronizing words of X and fix $m \in H(X)$. Suppose $(\!(x, m)\!)$ be as in (2.1). Suppose's assume that set M is equal to set

$$\{a \in W_n(X) : \exists (\!(x, m)\!) \text{ s.t. } mam = x_{[-|mam|+1, 0]}\}.$$

Set

$$h(m, X) := \limsup_{n\to\infty} \frac{1}{n} \log |M| .$$

Next proposition shows that $h(m, X)$ is independent of m.

Theorem 3.1. *If m is an arbitrary half synchronizing word for X with X_0^+ its right Fischer cover, then*

$$h(m, X) = h(X_0^+). \tag{3.1}$$

Proof. Pick a left transitive $x \in X$ such that $x_{[-|m|+1, 0]} = m$ and $w_+(x_{(-\infty, 0]}) = w_+(m)$ and set $\theta := w_+(x_-)$, $\delta := w_+(\ldots x_{-|m|-1}x_{-|m|})$. Suppose π be a finite path in X_0^+ such that $i(\pi) = \theta$, $t(\pi) = \delta$, $\mathcal{L}(\pi) = b$. Suppose τ is an arbitrary cycle from θ to θ in X_0^+ with $\mathcal{L}(\tau) = a$. Set $z_- := x_-abm$. Then, z is left transitive and $w_+(z_-) = w_+(m)$. Thus the number of cycles of length $|\tau|$ from θ to θ is at most $|S_n|$ where

$$S_n = \{u \in W_n(X) : \exists (\!(x, m)\!) \ni mum = x_{[-|mum|+1, 0]}\}$$

and $n = |\tau| + |b|$. This means $h(X_0^+) \le h(m, X)$.

Conversely, suppose $\epsilon > 0$ and pick $\{n_k : k \in \mathbb{N}\}$ such that $n_k < n_{k+1}$ and

$$h(m, X) - \epsilon < \lim_k \frac{1}{n_k} \log |S_{n_k}| \le h(m, X) \tag{3.2}$$

where $S_{n_k} = \{a_1^k, \ldots, a_{j_k}^k\}$ For each $k \in \mathbb{N}$ and

$$1 \le i \le j_k.$$

Choose $(\!(x^{(ik)}, m)\!)$ such that

$$x^{(ik)}_{[-|ma_i^k m|+1, 0]} = ma_i^k m.$$

Set $\theta_i^k := w_+(x_-^{(ik)})$. We will have

$$w_+(m) = w_+(x_-^{(ik)}) = w_+(\theta_i^k)[3, Page 146].$$

Thus $w_+(\theta_1^k) = w_+(\theta_2^k) = \cdots = w_+(\theta_{j_k}^k)$ and so $\theta_1^k = \theta_2^k = \cdots = \theta_{j_k}^k = w_+(m)$. Since $w_+(x_-^{(ik)} a_i^k m) = w_+(x_-^{(ik)})$, therefore there are cycles C_i^k in X_0^+ labeled $a_i^k m$ and initiating at $\theta_1^1 = w_+(m)$. Thus j_k is at most as large as the number of cycles of length $n_k + |m|$ based at $\theta_1^1 = w_+(m)$. Set

$$H_1 = C_1^1 \cup \ldots \cup C_{j_1}^1, \ldots, H_k = H_{k-1} \cup C_1^k \cup \ldots \cup C_{j_k}^k.$$

Then, $\lim_k h(H_k) \ge \lim_k \frac{1}{n_k} \log |S_{n_k}|$ and so by (3.2), there is $k \in \mathbb{N}$ such that

$$h(m, X) - \epsilon < h(H_k). \tag{3.3}$$

But H_k is a subgraph of X_0^+. Thus $h(H_k) \le h(X_0^+)$ and therefore by (3.3), $h(m, X) - \epsilon < h(X_0^+)$. This means that $h(m, X) \le h(X_0^+)$.

□

Define the *half synchronized entropy* $h_{\text{hsyn}}(X)$ to be

$$h_{\text{hsyn}}(X) := h(m, X), \tag{3.4}$$

where m is an arbitrary half synchronizing word for X. By the above proposition, (3.4) is well defined. Moreover, if X is synchronized, then $h_{\text{hsyn}}(X) = h_{\text{syn}}(X)$.

Theorem 3.2. *Suppose $X = R(X)$. Then, for any synchronizing word such as m, there is a unique "magic" vertex in X_0^+ which is the terminal of any path labeled m. Now if $x \in R(X)$ have infinitely many synchronizing words is past and future, then there must be a bi-infinite path labeled x, say π_x in X_0^+ passing through m and the magic vertex.*

By the above remark, an equivalent statement is

$$\sup_{i\in\mathbb{Z}}\{I_x\} \tag{3.5}$$

where

$$I_x = \{\inf\{(j-i) \geq 0 : \; x_{[i,j]} \in S(X),\, \pi_x \in X_0^+\}.$$

Recall that $S(X)$ is the synchronized word of X.

In the next section, we use a definition similar to (3.5) to extend the results to half-synchronizing case.

In this part we intend to extend synchronizing entropy to half synchronized systems, therefore suppose X be half synchronized system. One may use the same routine as in the synchronized case by applying some adjusments. For instance, for say synchronizing entropy, can be replaced by $\sup_{i\in\mathbb{Z}} i_x$ where

$$i_x = \{\inf\{(j-i) \geq 0 : \; w_+(x_{(-\infty,j]}) = w_+(x_{[i,j]})\}\} \tag{3.6}$$

where again $\pi_x \in X_0^+$ is a path labeled x. Note that

$$w_+(x_{(-\infty,j]}) = w_+(x_{[i,j]}), \quad \pi_x \in X_0^+ \tag{3.7}$$

means that $x_{[i,j]}$ is a half synchronizing word.

However, due to irreducibility and therefore the existence of Fischer cover for X, we will choose another proof which we think it gives a better picture.

Fix $m \in H(X)$. Then there is a unique vertex I in X_0^+ such that $w_+(m) = w_+(I) := \{\mathcal{L}(\pi) : i(\pi) = I\}$. For every $n \in \mathbb{N}$, define C_n to be the set of all cycles C in X_0^+ starting at $w_+(m)$ such that $|C| = n$ and $m \subseteq \mathcal{L}(C)$.

Set

$$H_1 := \bigcup_{C\in C_1} C, \, \ldots, \, H_n := H_{n-1} \cup \left(\cup_{C\in C_{n-1}} C\right)$$

and $Z_m := \bigcup_n X_{H_n}$. Observe that $\overline{Z_m} = X$. Also all $C \in C_n$ meet at $w_+(m)$ and therefore X_{H_n} is an irreducible sofic.

Theorem 3.3. *Suppose X be a half sychronized system. Then, there is a sequence $A_1 \subseteq A_2 \subseteq \cdots$ of irreducible shift of finite type's in X such that $\lim_n h(A_n)$ is equal to*

$$\sup\{h(A) : \; A \subseteq Z_m \;\; \textit{is an irreducible shift of finite type}\}. \tag{3.8}$$

Also

$$\lim_n h(A_n) = h(X_0^+) = h_{hsyn}(\overline{Z_m}).$$

Proof. Set

$$t_0 := \sup\{h(A) : A \subseteq Z_m \text{ is an irreducible shift of finite type}\}$$

and suppose $A \subseteq Z_m$ be an irreducible shift of finite type. Then, $h(A) = h_{\text{syn}}(A) \leq h(X_0^+)$ and this implies $t_0 \leq h(X_0^+)$.

To prove $h(X_0^+) \leq t_0$, fix $l \in \mathbb{N}$. There is $\{l_k : \; k \in \mathbb{N}\}$ such that $l_k < l_{k+1}$ and

$$h(X_0^+) - \frac{1}{l} < \lim_k \frac{1}{l_k} \log |C_{l_k}| \leq h(X_0^+) \tag{3.9}$$

where C_{l_k} is as in C_n. Then, $\lim_k h(X_{H_{l_k}}) \geq \lim_k \frac{1}{l_k} \log |C_{l_k}|$. To save the notation, we may assume that $\lim_k h(X_{H_k}) \geq \lim_k \frac{1}{k} \log |C_k|$ and therefore by (3.9), there is $k_j \in \mathbb{N}$ such that

$$h(X_0^+) - \frac{1}{l} < h(X_{H_{k_j}}). \tag{3.10}$$

$X_{H_{k_j}} \subseteq Z_m$ is an irreducible sofic and by [9, Theorem 3.2], there is a sequence $A_1 \subseteq A_2 \subseteq \cdots$ of irreducible shift of finite type's in $X_{H_{k_j}}$ such that

$$\lim_i h(A_i) = h\left((X_{H_{k_j}})_0^+\right) = h_{\text{syn}}(X_{H_{k_j}})$$

and moreover as a sofic,

$$h_{\text{syn}}(X_{H_{k_j}}) = h(X_{H_{k_j}})[10, Lemma 3.1].$$

Thus $\lim_i h(A_i) = h(X_{H_{k_j}})$ and therefore by (3.10), there is $i_l \in \mathbb{N}$ such that $h(X_0^+) - \frac{1}{l} < h(A_{i_l})$. Thus , $h(X_0^+) - \frac{1}{l} < t_0$ and therefore $h(X_0^+) \leq t_0$ as required and as a result $\lim_l h(A_{i_l}) = t_0$. By setting $A_n := A_{i_l}$, we will have (3.8).

□

Theorem 3.4. *If X is half synchronized and $h_{hsyn}(X) = h(X)$, then X is almost sofic.*

Also note that if a general subshift X has a half synchronized component say Y which is not synchronized, then $Y \subseteq \partial^n X$ for $n \in \mathbb{N} \cup \{0\}$. In other words, if Z is a subsystem of X such that $Z \nsubseteq \cap_n \partial^n X$, then Z is a half synchronized component of X if and only if Z is a synchronized component of X.

4 Conclusion

A word m is weak synchronizing, when there is a a left ray x_- such that if x_-m and mw are admissible, then x_-mw is also admissible. The respective subshifts are called weak synchronized. We, using a rather different approach, to show that this result extends to weak synchronized systems. In exploring the properties of weak synchronizing words and their corresponding subshifts, we uncover a rich structure that highlights the interplay between word formation and admissibility. To illustrate this concept, consider any weakly synchronized system defined by the presence of such a left ray. When we examine the extension, we observe that the admissibility conditions associate the behavior of with that of and. This relationship evokes thoughts of closure properties inherent in subshift dynamics, where concatenation maintains structural constraints that dictate admissible sequences. Consequently, this setup allows us to formulate general principles governing the extension of words in weakly synchronized systems. Moreover, the methodology employed in this study diverges from traditional approaches, positing that the pathways towards establishing weak synchronization can reveal new avenues for understanding dynamic properties of symbolic systems. By approaching the concept of weak synchronization from this fresh perspective, we aim to lay the groundwork for future exploration into associated phenomena, potentially unraveling complexities in more elaborate systems. In conclusion, our findings suggest that weak synchronized systems harbor fascinating structural characteristics that intricately link words and admissibility. As we delve deeper into these connections, we anticipate the emergence of novel insights that will enrich the discourse surrounding symbolic dynamics and contribute to the broader field of automata theory.

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References

[1] D.A. Dastjerdi, M. Shahamat, Kreiger graphs and Fischer covers vs dynamical properties. *J. Adv. Math. Comput. Sci*, **32**, 1–12, (2019).

[2] D. Lind, B. Marcus, An introduction to symbolic dynamics and coding. *Cambridge Univ*, Press. (1995).

[3] D. Fiebig, U. Fiebig, Covers for coded systems. *Contemporary Mathematics*, **135**, 139–179, (1992).

[4] S.J. Shaldehi, Suberesolving codes. *Tamking Journal of Mathematics*, **54:3**, 257–264, (2023).

[5] T. Meyerovitvh, Tail invariant measures of the Dyke-shift and non-sofic systems. *M.Sc. Thesis, Tel-Aviv university*, (2004).

[6] U. Jung, On the existence of open and bi-continuous codes. *Trans. Amer. Math. Soc.*, **363**, 1399–1417, (2011).

[7] M. Shahamat, Synchronized components of a subshift. *J. Korean Math. Soc.*, **59:1**, 1–12, (2022).

[8] M. Shahamat, Strong Synchronized System. *Journal of Mathematical Extension*, **16:6**, 1–16, (2022).

[9] K. Thomsen, On the ergodic theory of synchronized systems. *Ergod. Th. Dynam. Sys.*, **356**, 1235–1256, (2006).

[10] K. Thomsen, On the structure of a sofic subshift. *American Mathematical Society*, **356:9**, 3557–3619 (2004).

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