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ON NONLINEAR PROBLEM FOR THE THIRD ORDER EQUATION

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Abstract: In this paper, we examine the finite-time behavior of the solution to a mixed problem concerning the equation of third-order nonlinearity in its principal part. By employing Levine's lemma for a function that depends intricately on the solution of the initial-boundary value problem and its derivatives with respect to both x and t, we derive sufficient conditions for the blow-up of this solution within a finite period of time. Our investigation uncovers the nuanced dynamics at play, highlighting the delicate interplay between nonlinearity and time within the context of the problem. Through rigorous analysis, we aim to illuminate the conditions under which solutions may exhibit singular behavior, contributing to a deeper understanding of the complexities inherent in nonlinear differential equations. This exploration not only enriches the theoretical framework surrounding such problems but also sets the stage for potential applications in various fields that grapple with similar mathematical phenomena.

Keywords and phrases: Behavior for the solution, nonlinearity, pseudohyperbolic, blow-up of the solution, Levin's lemma

MSC 2010 Classifications: Primary 35L75; Secondary 35L82.

1 Introduction

The study of nonlinear boundary conditions has opened new avenues in the analysis of pseudoparabolic and pseudohyperbolic equations, revealing complex dynamics that were previously overlooked in the linear framework. Researchers have begun to investigate the impact of these nonlinearities on the existence, uniqueness, and regularity of solutions. Notably, the work [5] demonstrates that nonlinearities can significantly influence the qualitative behavior of solutions, leading to phenomena such as blow-up or the loss of smoothness under certain conditions. Furthermore, the exploration of mixed boundary conditions has gained traction, prompting a reevaluation of classical approaches. In particular, cases where Dirichlet and Neumann conditions interplay in nonlinear contexts have raised intriguing questions about stability and bifurcation. The coupling of these boundary conditions forces a deeper understanding of the interaction between the solutions and the domain's geometry, leading to a richer theoretical framework. In light of these developments, ongoing research continues to address the numerical aspects of these equations. The implementation of advanced computational techniques is essential for capturing the intricate patterns that arise from nonlinear boundary conditions. Approximative methods and simulations serve as crucial tools for validating theoretical predictions and for exploring parameter spaces where traditional analytical techniques falter. Ultimately, as the field evolves, addressing the challenges posed by nonlinear Dirichlet and Neumann boundary conditions will remain a central focus. The implications of these studies extend beyond pure mathematics, influencing applied domains such as fluid dynamics, materials science, and biological systems, underscoring the interdisciplinary nature of current mathematical research.

To begin our investigation, we define the mixed problem more rigorously, specifying the boundary conditions that govern the behavior of our system. The presence of third-order nonlinearity introduces complexities that warrant careful handling, particularly in terms of the regularity and continuity of solutions. We identify an appropriate function space, incorporating both spatial and temporal dimensions, to rigorously analyze the implications of our boundary constraints. Using Lemma Levine, we establish a framework for evaluating the growth rates of solutions in association with their derivatives. This framework enables us to express the conditions under which the solutions may exhibit blow-up phenomena. By examining the derived inequalities, we ascertain the relationship between initial conditions and the eventual behavior of the solutions. Further, we explore the implications of various parameter configurations, showcasing how changes in nonlinearity influence the solution's stability. The resultant analysis not only sheds light on critical thresholds that necessitate consideration but also assists in establishing a more comprehensive understanding of the dynamics involved. Ultimately, the findings underscore the delicate interplay between initial conditions and mixed boundary conditions in exacerbating or mitigating blow-up scenarios.

2 Formulation of the problem

Consider the following problem

$$u_{tt} - \sum_{i=1}^{n} D_i(|D_i u|^{p-2} D_i u) - \alpha\Delta u_t + f(u) = 0, \quad (x,t) \in \Omega \times [0,T] \tag{2.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_0(x), \quad x \in \Omega, \tag{2.2}$$

$$\sum_{i=1}^{n}(|D_i u|^{p-2} D_i u)\cos(x_i, \upsilon) + \alpha\frac{\partial u_t}{\partial \upsilon} = g(u), \quad (x,t) \in \partial\Omega \times [0,T], \tag{2.3}$$

where $\Omega \subset R^n$, $n \geq 2$ is bounded domain with smooth boundary $\partial\Omega$, $u_0(x) \in W_2^1(\Omega)$, $u_1(x) \in L_2(\Omega)$ are given functions, $f(u)$ and $g(u)$- are some nonlinear functions, $\alpha > 0$-are some number, $p \geq 2$, $D_i = \dfrac{\partial}{\partial x_i}$, $i = 1, 2, ..., n$, $\Delta = \sum\limits_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}$-is Laplace operator, $\dfrac{\partial}{\partial n}$-is derivative on the external normal at $\partial\Omega$.

Still now, problems with linear and homogeneous Dirichlet and Neumann boundary conditions for various pseudoparabolic and pseudohyperbolic equations have been well studied (see [1]-[8] and the literature cited therein). In recent years, results have been obtained when the boundary conditions are nonlinear (for example, in [5], the question of the destruction of solutions for the Aller equation was considered in the case when $p = 2$).

In this paper, we study the question of blow up of solutions for a nonlinear pseudohyperbolic equation for$p \geq 2$. For the problem (2.1)-(2.3) it is proved the following theorem

3 Solutions of a mixed problem for a third-order equation

Theorem 3.1. *Let for any $u \in R^1$ and for the functions $f(t)$ and $g(\tau)$ satisfies following conditions*

$$2(2\alpha+1)F(u) - uf(u) \geq 0, \quad F(u) = \int_0^u f(s)ds, \tag{3.1}$$

$$ug(u) - 2(2\alpha+1)G(u) \geq 0, \quad G(u) = \int_0^u g(\tau)d\tau, \tag{3.2}$$

$$\alpha \geq \frac{p-2}{4}. \tag{3.3}$$

Then if $\int\limits_{\Omega} F(u_0)dx - \int\limits_{\partial\Omega} G(u_0)ds + \frac{1}{p}\int\limits_{\Omega}\sum\limits_{i=1}^{n}|D_i u_o|^p\,dx \leq 0$, $(u_0, u_1) > 0$ *and for every solution* $u(x,t) \in W_2^1(0,T; W_2^2(\Omega)) \bigcap W_2^2(0,T; L_2(\Omega))$ *of the problem* (2.1)-(2.3)*, there exists* $t_0 < \infty$ *such that*

$$\lim_{t\to t_0}\left[\|u(x,t)\|^2 + \alpha\int_0^t \|\nabla(x,\tau)\|^2\,d\tau\right] = \infty.$$

Proof. Multiply both sides of the equation by u_t and integrate over the domain Ω, we get

$$\frac{1}{2}\frac{d}{dt}\left\|u_t\right\|^2-\int\limits_{\Omega}\sum_{i=1}^{n}D_i(\left|D_iu\right|^{p-2}D_iu)u_tdx-\int\limits_{\Omega}\Delta u_tu_tdx+\int\limits_{\Omega}g(u)u_tdx=0.$$

Applying integration by parts to the second and third terms and taking into account condition (2.3) and notations (3.1) and (3.2), after simple transformations we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\left\|u_t\right\|^2+\frac{1}{p}\int\limits_{\Omega}\sum_{i=1}^{n}\left|D_iu\right|^p dx+\int\limits_{\Omega}F(u)dx-\int\limits_{\partial\Omega}G(u)ds\right]+\int\limits_{\Omega}(\nabla u_t)^2dx=0, \quad (3.4)$$

Let's denote by

$$E(t)=\left\|u_t\right\|^2+\frac{1}{p}\int\limits_{\Omega}\sum_{i=1}^{n}\left|D_iu\right|^p dx+\int\limits_{\Omega}F(u)dx-\int\limits_{\partial\Omega}G(u)ds.$$

Then equality (3.4) can be we rewritten as

$$\frac{d}{dt}E(t)+\left\|\nabla u_t\right\|^2=0, \quad (3.5)$$

Integrating equality (3.5) over from zero to t, we obtain

$$E(t)-E(0)+\int\limits_0^t\left\|\nabla u_\tau(\tau)\right\|^2d\tau=0, \quad (3.6)$$

If we take into account the value of $E(t)$, and after moving $E(0)$the equality to the right side, we get

$$\frac{1}{2}\left\|u_t\right\|^2+\frac{1}{p}\int\limits_{\Omega}\sum_{i=1}^{n}\left|D_iu\right|^p dx+\int\limits_{\Omega}F(u)dx-\int\limits_{\partial\Omega}G(u)ds+\int\limits_0^t\left\|\nabla u_\tau(\tau)\right\|^2d\tau=M(0), \quad (3.7)$$

where

$$E(0)=\frac{1}{2}\left\|u_0\right\|^2+\frac{1}{p}\int\limits_{\Omega}\sum_{i=1}^{n}\left|D_iu_0\right|^p dx+\int\limits_{\Omega}F(u_0)dx-\int\limits_{\partial\Omega}G(u_0)ds.$$

Consider the function

$F(t)=\left\|u(x,t)\right\|^2+\int\limits_0^t\left\|\nabla u(x,\sigma)\right\|d\sigma+(l-t)\left\|\nabla u_0\right\|^2$, where l is some sufficiently large number.

Obviously

$$F(t)=2(u,u_t)+2\int\limits_0^t(\nabla u,\nabla u_\tau)d\tau,$$

$$F''(t)=2\left\|u_t\right\|^2+2(u,u_{tt})+2(\nabla u,\nabla u_\tau).$$

Taking into account here that, u is the solution to problem (2.1)-(2.3), we obtain

$$F''(t)=2\left\|u_t\right\|^2+2(u,u_{tt})+$$
$$+2(u,\sum_{i=1}^{n}D_i(\left|D_iu\right|^{p-2}D_iu))+2(u,\Delta u_t)-2(u,f(u))+2(\nabla u,\nabla u_t).$$

Applying integration by parts to the second and third terms and using condition (2.3), we have

$$F^{''}(t)=2\left\|u_t\right\|^2+2\int\limits_{\partial\Omega}\sum_{i=1}^{n}\left|D_iu\right|^{p-2}D_iu)\cos(x_i,n)ds-$$

$$-2\int\limits_{\Omega}\sum_{i=1}^{n}\left|D_iu\right|^{p-2}(D_iu)^2dx+2\int\limits_{\partial\Omega}\frac{\partial u_t}{\partial n}uds-2(\nabla u,\nabla u_t)-$$

$$-2(u,f(u)+2(\nabla u,\nabla u_t)=2\left\|u_t\right\|^2+2\int\limits_{\partial\Omega}(\sum_{i=1}^{n}\left|D_iu\right|^{p-2}D_iu)\cos(x_i,n)+\frac{\partial u_t}{\partial n})uds-$$

$$-2\int\limits_{\Omega}\sum_{i=1}^{n}\left|D_iu\right|^{p}dx-2(u,f(u)=2\left\|u_t\right\|^2+$$

$$+2\int\limits_{\partial\Omega}ug(u)ds-2\int\limits_{\Omega}uf(u)dx+2\int\limits_{\Omega}\sum_{i=1}^{n}\left|D_iu\right|^{p}dx.$$

Thus

$$F^{''}(t)=2\left\|u_t\right\|^2+2\int\limits_{\partial\Omega}ug(u)ds-2\int\limits_{\Omega}uf(u)dx+2\int\limits_{\Omega}\sum_{i=1}^{n}\left|D_iu\right|^{p}dx. \tag{3.8}$$

Multiplying both sides of equality (3.6) by $4(2\beta+1)$, we get

$$2(2\beta+1)\left\|u_t\right\|^2+$$

$$+\frac{4(2\beta+1)}{p}\int\limits_{\Omega}\sum_{i=1}^{n}\left|D_iu\right|^{p}dx-4(2\beta+1)\int\limits_{\partial\Omega}G(u)ds+4(2\beta+1)\int\limits_{\Omega}F(u)dx+$$

$$+4(2\beta+1)\int\limits_{0}^{t}\left\|\nabla u_\tau(\tau)\right\|d\tau=4(2\beta+1)E(0)2\int\limits_{0}^{t}(\nabla u,\nabla u_\tau)d\tau.$$

Let's rewrite this equality in the form

$$0=2(2\beta+1)\left\|u_t\right\|^2+$$

$$+\frac{4(2\beta+1)}{p}\int\limits_{\Omega}\sum_{i=1}^{n}\left|D_iu\right|^{p}dx-4(2\beta+1)\int\limits_{\partial\Omega}G(u)ds+4(2\beta+1)\int\limits_{\Omega}F(u)dx+$$

$$+4(2\beta+1)\int\limits_{0}^{t}\left\|\nabla u_\tau(\tau)\right\|d\tau=4(2\beta+1)E(0).$$

Let's add this equality to (3.8), after which we get

$$F^{''}(t) = 4(2\beta+1)\left\|u_t\right\|^2 + 4(2\beta+1)\int_0^t \left\|\nabla u_\tau(\tau)\right\| d\tau +$$

$$+2\int_{\partial\Omega}(ug(u) - 2(2\beta+1)G(u))ds + 2\int_{\Omega}(2(2\beta+1)F(u) - uf(u))dx +$$

$$+\left(\frac{4(2\beta+1)}{p} - 2\right)\int_{\Omega}\sum_{i=1}^{n}\left|D_i u\right|^p dx - 4(2\beta+1)E(0).$$

Taking into account conditions (3.1)-(3.3) of the theorem here, we will have

$$F^{''}(t)(t) \geq 4(\beta+1)\left[\left\|u_t\right\|^2 + \int_0^t \left\|\nabla u_\tau(\tau)\right\| d\tau\right].$$

Given the values $F(t)$and $F^{'}(t)$ the estimates $F^{''}(t)$ in the difference

$$F(t)F^{''}(t)) - (1+\beta)(F^{'}(t))^2$$

after some transformations we get

$$F(t)F^{''}(t)) - (1+\beta)(F^{'}(t))^2 \geq 0.$$

Then, by Lemma [2], we can approve that there exists $t_0 = \dfrac{F(0)}{\beta F^{'}(0)}$such that, there is

$$\lim_{t\to t_0}\left[\left\|u(x,t)\right\|^2 + \alpha\int_0^t \left\|\nabla u_\tau(x,\tau)\right\|^2 d\tau\right] = \infty.$$

Theorem was proved. □

4 Conclusion

The study of nonlinear problems associated with third-order equations is a compelling area of mathematical analysis, particularly due to the complexity and diverse applications these equations present. Third-order differential equations often arise in the modeling of physical phenomena, such as fluid dynamics, beam deflection, and wave propagation. The nonlinear nature of these equations poses significant challenges in obtaining analytical solutions, necessitating the development of robust numerical techniques and innovative analytical methods. One of the pivotal aspects of dealing with nonlinear third-order equations is the exploration of boundary value problems. The uniqueness and existence of solutions in such contexts hinge on the boundary conditions imposed. Various methods, including perturbation techniques and fixed-point theorems, can be leveraged to derive approximate solutions or to analyze the stability of solutions. Additionally, the integration of qualitative theory provides insight into the behavior of solutions and the influence of nonlinearity on their properties. Moreover, technological advancements have enabled the use of computational tools to tackle these complex equations, allowing for simulations that were previously infeasible. This intersection of mathematics and computational science opens new avenues for research and application, especially in engineering and physics. As we continue to delve into these nonlinear problems, the potential for discovering novel solutions and theoretical insights remains vast and promising.

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