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ON THE REGULARIZATION OF THE CAUCHY PROBLEM FOR MATRIX FACTORIZATIONS OF THE HELMHOLTZ EQUATION IN $\mathbb{R}^m$

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Abstract: In this paper, we delve into the intricate challenge of extending solutions to the ill-posed Cauchy problem linked to matrix factorizations of the Helmholtz equation, set within both bounded and unbounded multidimensional domains. We presuppose the existence of a solution that is continuously differentiable throughout the entire closed domain, anchored by the specified Cauchy data. Given these conditions, we derive explicit formulas for the extension of this solution alongside a robust regularization method. Our proposed solutions encompass continuous approximations that faithfully conform to a predetermined error measure within the uniform metric, effectively replacing the original Cauchy data. Furthermore, this study offers an estimation of the stability of the solution to the Cauchy problem, framed within a classical context. Through this exploration, we not only aim to advance the mathematical understanding of the Helmholtz equation but also to illuminate pathways for practical applications wherein such extended solutions can be effectively utilized.

Keywords and phrases: The Cauchy problem, regularization, factorization, regular solution, fundamental solution.

MSC 2010 Classifications: Primary 35J46; Secondary 35J56.

1 Introduction

The exploration of regularizing operators is crucial in the context of the Cauchy problem for the Helmholtz equation, as it facilitates the stabilization of solutions that are otherwise sensitive to perturbations in data. Regularization techniques, by incorporating prior knowledge or additional constraints, can effectively mitigate the inherent instability often present in ill-posed problems. Notably, the Carleman estimate serves as a powerful tool in deriving bounds for solutions and establishing the existence of regularizing operators. These estimates not only enhance our understanding of solution behavior but also lead to effective numerical schemes for solving complex inverse problems. In recent years, advancements in computational algorithms have significantly transformed our approach to these challenges. The iterative methods rooted in the Kozlov-Maz'ya-Fomin framework have shown remarkable success in practical applications, providing robust strategies for regularization that improve convergence rates. Leveraging such algorithms enables practitioners to obtain approximate solutions that adhere closely to the underlying physical phenomena, thereby enhancing predictive accuracy. Furthermore, interdisciplinary collaborations draw upon insights from applied mathematics and computational physics, promoting innovative methodologies tailored to various scientific domains. As the frontiers of research in matrix factorizations continue to expand, understanding the interplay between theoretical constructs and practical applications remains essential for tackling the complexities associated with the Helmholtz equation and its myriad implications across disciplines. The complexity of the Cauchy problem for elliptic equations arises from the interplay between the uniqueness of solutions and the characteristics of the data defined on an incomplete boundary. In contrast to Fredholm equations, where one can leverage well-established methods from functional analysis, the elliptic setting necessitates a more nuanced approach owing to the non-closed nature of the dense set of solvable data. This results in significant challenges for proving existence and regularity, leading to the necessity for innovative techniques and advanced tools that can operate within this less structured framework. Furthermore, the stability of solutions relies heavily on the geometric and analytical properties of the underlying domain and the operators involved. The works of Aizenberg, Kytmanov, and Tarkhanov provided critical advancements, establishing foundational results that explore stability conditions and perturbation theories specific to elliptic equations. Their contributions laid the groundwork for subsequent investigations into the robustness of solutions amid varying data configurations, enhancing the theoretical understanding of inverse problems. In particular, the incorporation of compactness into the solution space, as suggested by Tikhonov, aids in addressing the instability that arises from non-closed data sets. This restriction allows researchers to apply compactness arguments to derive well-posedness in particular scenarios, illuminating pathways for resolving the broader challenges posed by the elliptic Cauchy problem. The interplay of uniqueness, stability, and data density continues to be an area ripe for exploration, promising rich avenues for future research (see, for instance [1, 18]).

The regularization techniques employed in the context of hyperbolic equations typically involve the introduction of additional parameters or constraints that allow for more tractable solutions. These techniques can be particularly significant in problems where classical solutions may become ill-defined or exhibit discontinuities. For instance, using smoothing functions or convolution operators can lead to well-defined approximations that effectively capture the underlying dynamics of the system, allowing one to analyze the behavior of solutions even in challenging scenarios. Moreover, the adaptability of regularization formulas is a vital aspect of their utility; they can be calibrated according to the specifics of the initial conditions or other influencing factors. This flexibility enhances their application across various domains, from fluid dynamics to wave propagation, encouraging researchers to devise customized approaches tailored to the peculiarities of their particular problems. As a result, the theoretical groundwork laid in exploring Cauchy problems becomes invaluable for practitioners seeking reliable computational techniques. In essence, the exploration of regularization formulas transcends mere theoretical investigation, fostering synergies between mathematical theory and practical application. With ongoing advancements in computational methods and numerical algorithms, the prospect of harnessing these regularization strategies in real-world scenarios remains promising, offering a pathway to overcoming challenges inherent in the absence of classical solutions.

This issue pertains to ill-posed problems, indicating a degree of instability. It is acknowledged that the Cauchy problem for elliptic equations exhibits instability in response to minor alterations in the data, which signifies its incorrectness (refer to Hadamard, for example, see [3], p. 39). There exists a substantial body of literature on this topic (see, for instance, [2]-[8]). In his work, N.N. Tarkhanov [15] introduced a criterion for determining the solvability of a broader category of boundary value problems concerning elliptic systems. In scenarios involving unstable problems, the operator's image is not closed. Consequently, the conditions for solvability cannot be expressed solely in terms of continuous linear functionals. Thus, within the Cauchy problem for elliptic equations that utilize data from a portion of the boundary, solutions typically exist uniquely. This problem is valid for a set of data that is dense everywhere, though this set is not closed. As a result, the theoretical framework for the solvability of these problems is far more complex and profound than that related to Fredholm equations. Initial findings in this area emerged in the mid-1980s through the contributions of L.A. Aizenberg, A.M. Kytmanov, and N.N. Tarkhanov (see, for example, [16]). The distinctiveness of the solution is derived from Holmgren's general theorem (refer to [7]), while the problem's conditional stability is established through the research of A.N. Tikhonov (see [1]), provided we limit the set of potential solutions to a compact space.

In pursuit of understanding these complexities, researchers have employed diverse mathematical tools and theoretical frameworks. The exploration of operator theory, particularly within the context of unbounded operators, plays a pivotal role. By identifying appropriate domains, one can establish the continuity and boundedness critical to yielding solvable mathematical models. Moreover, advancements in functional analysis have fostered significant developments, particularly in the realm of spectral theory, which offers insights into the behavior of solutions under various conditions and constraints. The interplay between regularization techniques and numerical methods has also garnered attention. Techniques such as Tikhonov regularization and Landweber iteration provide robust avenues for approximating solutions to ill-posed problems. By refining these approximations, researchers are now capable of tackling problems that previously appeared intractable. The Cauchy problem related to the Helmholtz equation, for example, exemplifies how these approaches can offer potential solutions that adhere to physical principles while addressing mathematical shortcomings. Furthermore, boundary value problems present a fertile ground for investigation. The complexities associated with non-standard boundary conditions often necessitate innovative computational strategies. Recent studies have highlighted the importance of adaptive meshes and iterative solvers, demonstrating their effectiveness in enhancing the accuracy of solutions across diverse geometries and conditions. As the frontier of mathematical physics continues to expand, collaboration across disciplines remains crucial in deciphering the myriad challenges that lie ahead.

The challenge of reconstructing solutions in this context lies in the inherent limitations imposed by the boundary conditions. When dealing with first-order elliptic systems, the boundary data often provides insufficient information for a unique solution across the entire domain. As a result, the inverse problem becomes crucial, where one seeks to determine the underlying behavior of the solution based on partial or indirect information. This scenario highlights the significance of various mathematical methods, such as operator theory and functional analysis, which can be employed to address the gaps in data and derive approximations of the desired solutions. Further complicating the situation is the nature of the Helmholtz operator itself. Its factorization involves understanding the interplay between differential operators and the associated boundary value problems. In many cases, the factorization leads to a simplification of the underlying equations, but it also raises new questions about the well-posedness and stability of the resulting solution. Researchers have explored various techniques to ensure that the solutions remain robust despite the restricted boundary conditions, often involving regularization methods and stability analysis. Moreover, recent advancements in the field of numerical methods have provided new avenues for tackling these reconstructive challenges. Computational approaches, particularly those based on iterative methods and state-of-the-art algorithms, have shown promise in recovering solutions from incomplete data. These methodologies not only enhance our understanding of elliptic systems but also expand their applicability across various scientific domains, from engineering to physics, where such systems are prevalent. Consequently, the ongoing investigation into these problems stands at the forefront of modern mathematical research, reflecting both their theoretical significance and practical implications.

The Cauchy problem associated with many elliptic equations has a singular solution, indicating that this problem can be addressed for a data set that is dense throughout the space, although this data set is not regarded as closed. Consequently, the study of the solvability of such problems is quite intricate. References [10]-[21] provide an in-depth exploration of the characteristics and attributes of these issues. In works [22]-[32], approximate solutions to the ill-posed Cauchy problem are analyzed through various factorizations of the Helmholtz operator. Building upon these findings, explicit regularized solutions for the Cauchy problem have been derived for different factorizations associated with the Helmholtz operator, as discussed in [33]-[49].

Let $\mathbb{R}^m$ be a $m-$dimensional real Euclidean space,

$$x=(x_1,\dots,x_m)\in\mathbb{R}^m, y=(y_1,\dots,y_m)\in\mathbb{R}^m,$$

$$x'=(x_1,\dots,x_{m-1})\in\mathbb{R}^{m-1}, y'=(y_1,\dots,y_{m-1})\in\mathbb{R}^{m-1},$$

We introduce the following notation:

$$r=|y-x|\,,\ \alpha=|y'-x'|\,,\ w=i\sqrt{u^2+\alpha^2}+y_m,\ w_0=i\alpha+y_m,\ u\geq 0,\ s=\alpha^2,$$

$$w=i\tau\sqrt{u^2+\alpha^2}+\beta,\ w_0=i\tau\alpha+\beta,\ \beta=\tau y_m,\ \tau=tg\frac{\pi}{2\rho},\ \rho>1,$$

$$G_\rho=\{y:\ |y'|<\tau y_m,\ y_m>0\},\ \partial G_\rho=\{y:\ |y'|=\tau y_m,\ y_m>0\},$$

$$\frac{\partial}{\partial x}=\left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_m}\right)^T,\ \frac{\partial}{\partial x}\to\xi^T,\ \xi^T=\begin{pmatrix}\xi_1\\ \dots\\ \xi_m\end{pmatrix}-\text{transposed vector }\xi,$$

$$U(x)=(U_1(x),\dots,U_n(x))^T,\ u^0=(1,\dots,1)\in\mathbb{R}^n,\ n=2^m,\ m\geq 2,$$

$$E(z)=\left\|\begin{array}{cccc} z_1 & 0 & \ldots & 0\\ 0 & z_2 & \ldots & 0\\ \ldots & \ldots & \ddots & \ldots\\ 0 & 0 & \ldots & z_n \end{array}\right\| -\text{diagonal matrix, } z=(z_1,\ldots,z_n)\in\mathbb{R}^n.$$

Let $D(\xi^T)$ be a $(n\times n)-$dimensional matrix with elements consisting of a set of linear functions with constant coefficients of the complex plane for which the following condition is satisfied:

$$D^*(\xi^T)D(\xi^T)=E((|\xi|^2+\lambda^2)u^0),$$

where $D^*(\xi^T)$ is the Hermitian conjugate matrix $D(\xi^T)$, λ is a real number.

We will consider the following system of linear first order partial differential equations

$$D\left(\frac{\partial}{\partial x}\right)U(x)=0, \tag{1.1}$$

where $D\left(\frac{\partial}{\partial x}\right)$ is the matrix of first-order differential operators.

$G_\rho\subset\mathbb{R}^m$ be a bounded simply-connected domain, the boundary of which consists of the surface of the cone ∂G_ρ, and a smooth piece of the surface S, lying in the cone G_ρ, i.e., $\partial G_\rho = S\bigcup T,\quad T=\partial G_\rho\backslash S.$

Let $(0,\ldots,x_m)\in G_\rho,\quad x_m>0.$

We denote by $A(G_\rho)$ the class of vector functions in the domain G_ρ continuous on $\overline{G}_\rho = G_\rho\bigcup\partial G_\rho$ and satisfying system (1.1).

Problem 1. Suppose $U(y)\in A(G_\rho)$ and

$$U(y)|_S=f(y),\quad y\in S. \tag{1.2}$$

Here, $f(y)$ a given continuous vector-function on S. It is required to restore the vector function $U(y)$ in the domainG_ρ, based on it's values $f(y)$ on S.

If $U(y)\in A(G_\rho)$, then the following integral formula of Cauchy type is valid

$$U(x)=\int\limits_{\partial G_\rho} N(y,x;\lambda)U(y)ds_y,\quad x\in G_\rho. \tag{1.3}$$

where

$$N(y,x;\lambda)=\left(E(\varphi_m(\lambda r)u^0)D^*\left(\frac{\partial}{\partial x}\right)\right)D(t^T).$$

Here $t=(t_1,\ldots,t_m)-$is the unit exterior normal, drawn at a point y, the surface ∂G_ρ, $\varphi_m(\lambda r)$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^m$, where $\varphi_m(\lambda r)$ defined by the following formula:

$$\begin{aligned}\varphi_m(\lambda r)&=P_m\lambda^{(m-2)/2}\frac{H^{(1)}_{(m-2)/2}(\lambda r)}{r^{(m-2)/2}},\\ P_m&=\frac{1}{2i(2\pi)^{(m-2)/2}}.\end{aligned} \tag{1.4}$$

Here $H^{(1)}_{(m-2)/2}(\lambda r)$ is the Hankel function of the first kind of $(m-2)/2-$th order (see, for instance [9]).

We denote by $K(w)$ is an entire function taking real values for real w,($w=u+iv$, $u,v-$real numbers) and satisfying the following conditions:

$$\begin{gathered}K(u)\neq 0,\quad \sup_{v\geq 1}\left|v^pK^{(p)}(w)\right|=B(u,p)<\infty,\\ -\infty<u<\infty,\quad p=0,1,...,m.\end{gathered} \tag{1.5}$$

We define the function $\Phi(y,x;\lambda)$ at $y\neq x$ by the following equalities

$$\Phi(y,x;\lambda) = \frac{1}{c_m K(x_m)} \frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^\infty \mathrm{Im}\left[\frac{K(w)}{w-x_m}\right] \frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}} du, \tag{1.6}$$
$$m=2k, \quad k \geq 1, \quad c_m = (-1)^{k-1}(k-1)!(m-2)\omega_m,$$

$$\Phi(y,x;\lambda) = \frac{1}{c_m K(x_m)} \frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^\infty \mathrm{Im}\left[\frac{K(w)}{w-x_m}\right] \frac{\cos(\lambda u)}{\sqrt{u^2+\alpha^2}} du, \tag{1.7}$$
$$m=2k+1, \quad k \geq 1, \quad c_m = (-1)^k 2^{-k}(2k-1)!(m-2)\pi\omega_m.$$

Where $I_0(\lambda u) = J_0(i\lambda u)$—is the Bessel function of the first kind of zero order, ω_m—area of a unit sphere in space $\mathbb{R}^m$.

In the formula (1.6) and (1.7), choosing

$$K(w) = E_\rho(\sigma^{1/\rho}w), \quad K(x_m) = E_\rho(\sigma^{1/\rho}\gamma), \quad \gamma = \tau x_m, \quad \sigma > 0. \tag{1.8}$$

we get

$$\Phi_\sigma(y,x;\lambda) = \frac{E_\rho^{-1}(\sigma^{1/\rho}\gamma)}{c_m} \frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^\infty \mathrm{Im}\left[\frac{E_\rho(\sigma^{1/\rho}w)}{w-x_m}\right] \frac{uI_0(\lambda u)}{\sqrt{u^2+\alpha^2}} du, \tag{1.9}$$
$$m=2k, \quad k \geq 1,$$

$$\Phi_\sigma(y,x;\lambda) = \frac{E_\rho^{-1}(\sigma^{1/\rho}\gamma)}{c_m} \frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^\infty \mathrm{Im}\left[\frac{E_\rho(\sigma^{1/\rho}w)}{w-x_m}\right] \frac{\cos(\lambda u)}{\sqrt{u^2+\alpha^2}} du, \tag{1.10}$$
$$m=2k+1, \quad k \geq 1.$$

Here $E_\rho(\sigma^{1/\rho}w)$ is the entire Mittag-Leffler function (see [7]).

The formula (1.3) is true if instead $\varphi_m(\lambda r)$ of substituting the function

$$\Phi_\sigma(y,x;\lambda) = \varphi_m(\lambda r) + g_\sigma(y,x;\lambda). \tag{1.11}$$

Then the integral formula (1.3) has the form:

$$U(x) = \int\limits_{\partial G_\rho} N_\sigma(y,x;\lambda)U(y)ds_y, \quad x \in G_\rho. \tag{1.12}$$

where

$$N_\sigma(y,x;\lambda) = \left(E(\Phi_\sigma(y,x;\lambda)u^0)D^*\left(\frac{\partial}{\partial x}\right)\right)D(t^T)$$

2 Solution of the Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain

The Cauchy problem for matrix factorizations of the Helmholtz equation presents a complex challenge in applied mathematics and computational physics, particularly within the context of multidimensional bounded domains. This problem involves determining the unknown parameter fields of the equation from incomplete or noisy measurements, a situation often encountered in fields such as geophysics and biomedical imaging. The Helmholtz equation itself, characterized by the wave nature of solutions, necessitates careful consideration of boundary conditions and matrix structures that arise in its factorization approaches. Exploring the interplay between linear algebra and numerical analysis is essential for understanding the intricate behavior of the solutions to the Helmholtz equation. Recent advancements in computational power and algorithms have further enhanced our ability to simulate and solve these types of problems effectively, paving the way for innovative applications across various scientific domains.

Theorem 2.1. *Let $U(y) \in A(G_\rho)$ it satisfy the inequality*

$$|U(y)| \leq M, \quad y \in T = \partial G_\rho \backslash S, \quad x \in G_\rho. \tag{2.1}$$

If

$$U_\sigma(x) = \int\limits_S N_\sigma(y, x; \lambda) U(y) ds_y, \quad x \in G_\rho, \tag{2.2}$$

then the following estimates are true:
at $m = 2k, \quad k \geq 1$:

$$|U(x) - U_\sigma(x)| \leq C_\rho(\lambda, x) M \sigma^k \exp(-\sigma\gamma^\rho), \quad \sigma > 1, \quad x \in G_\rho, \tag{2.3}$$

$$\left| \frac{\partial U(x)}{\partial x_j} - \frac{\partial U_\sigma(x)}{\partial x_j} \right| \leq C_\rho(\lambda, x) M \sigma^k \exp(-\sigma\gamma^\rho), \quad \sigma > 1, \quad x \in G_\rho, \quad j = \overline{1, m}, \tag{2.4}$$

at $m = 2k + 1, \quad k \geq 1$:

$$|U(x) - U_\sigma(x)| \leq C_\rho(x) M \sigma^{k+1} \exp(-\sigma\gamma^\rho), \quad \sigma > 1, \quad x \in G_\rho, \tag{2.5}$$

$$\left| \frac{\partial U(x)}{\partial x_j} - \frac{\partial U_\sigma(x)}{\partial x_j} \right| \leq C_\rho(x) M \sigma^{k+1} \exp(-\sigma\gamma^\rho), \quad \sigma > 1, \quad x \in G_\rho, \quad j = \overline{1, m}. \tag{2.6}$$

Here and below functions bounded on compact subsets of the domain G_ρ, we denote by $C_\rho(\lambda, x)$ and $C_\rho(x)$.

Corollary 2.2. *For each $x \in G_\rho$ the equalities are true*

$$\lim_{\sigma \to \infty} U_\sigma(x) = U(x), \quad \lim_{\sigma \to \infty} \frac{\partial U_\sigma(x)}{\partial x_j} = \frac{\partial U(x)}{\partial x_j}, \quad j = \overline{1, m}.$$

We denote by $\overline{G}_\varepsilon$ the set

$$\overline{G}_\varepsilon = \{(x_1, \ldots, x_m) \in G_\rho, \quad a > x_m \geq \varepsilon, \quad a = \max_T \psi(x'), \quad 0 < \varepsilon < a\}.$$

Here, at $m = 2, \quad \psi(x_1) -$is a surface, and at $m > 2, \quad \psi(x') -$is a surface. It is easy to see that the set $\overline{G}_\varepsilon \subset G_\rho$ is compact.

Corollary 2.3. *If $x \in \overline{G}_\varepsilon$, then the families of functions $\{U_\sigma(x)\}$ and $\left\{ \frac{\partial U_\sigma(x)}{\partial x_j} \right\}$ converge uniformly for $\sigma \to \infty$, i.e.:*

$$U_\sigma(x) \rightrightarrows U(x), \quad \frac{\partial U_\sigma(x)}{\partial x_j} \rightrightarrows \frac{\partial U(x)}{\partial x_j}, \quad j = \overline{1, m}.$$

Suppose that the curve S is given by the equation

$$y_m = \psi(y'), \quad y' \in \mathbb{R}^{m-1}$$

where $\psi(y')$ is a single-valued function satisfying the Lyapunov conditions.
We put

$$a = \max_T \psi(y'), \quad b = \max_T \sqrt{1 + \psi'^2(y')}.$$

Theorem 2.4. *Let $U(y) \in A(G_\rho)$ satisfy condition (1.9), and on a smooth curve S the inequality*

$$|U(y)| \le \delta, \quad 0 < \delta < M, \quad y \in S. \tag{2.7}$$

Then the following estimates is true

at $m = 2k, \quad k \ge 1$:

$$|U(x)| \le C_\rho(\lambda, x)\sigma^k M^{1-\left(\frac{\gamma}{a}\right)^\rho} \delta^{\left(\frac{\gamma}{a}\right)^\rho}, \quad \sigma > 1, \quad x \in G_\rho, \tag{2.8}$$

$$\left|\frac{\partial U(x)}{\partial x_j}\right| \le C_\rho(\lambda, x)\sigma^k M^{1-\left(\frac{\gamma}{a}\right)^\rho} \delta^{\left(\frac{\gamma}{a}\right)^\rho}, \quad \sigma > 1, \quad x \in G_\rho, \quad j = \overline{1, m}. \tag{2.9}$$

at $m = 2k+1, \quad k \ge 1$:

$$|U(x)| \le C_\rho(x)\sigma^{k+1} M^{1-\left(\frac{\gamma}{a}\right)^\rho} \delta^{\left(\frac{\gamma}{a}\right)^\rho}, \quad \sigma > 1, \quad x \in G_\rho, \tag{2.10}$$

$$\left|\frac{\partial U(x)}{\partial x_j}\right| \le C_\rho(x)\sigma^{k+1} M^{1-\left(\frac{\gamma}{a}\right)^\rho} \delta^{\left(\frac{\gamma}{a}\right)^\rho}, \quad \sigma > 1, \quad x \in G_\rho, \quad j = \overline{1, m}. \tag{2.11}$$

Here is $a^\rho = \max_{y \in S} \operatorname{Re} w_0^\rho$.

Let $U(y) \in A(G_\rho)$ and instead of functions $U(y)$ on S with its approximations $f_\delta(y)$respectively, with an error $0 < \delta < M$,

$$\max_S |U(y) - f_\delta(y)| \le \delta. \tag{2.12}$$

We put

$$U_{\sigma(\delta)}(x) = \int_S N_\sigma(y, x; \lambda) f_\sigma(y) ds_y, \quad x \in G_\rho. \tag{2.13}$$

Theorem 2.5. *Let $U(y) \in A(G_\rho)$ on the part of the plane $y_m = 0$ satisfy condition (1.9)*

Then the following estimates is true

at $m = 2k, \quad k \ge 1$:

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \le C_\rho(\lambda, x)\sigma^k M^{1-\left(\frac{\gamma}{a}\right)^\rho} \delta^{\left(\frac{\gamma}{a}\right)^\rho}, \quad \sigma > 1, \quad x \in G_\rho, \tag{2.14}$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right| \le C_\rho(\lambda, x)\sigma^k M^{1-\left(\frac{\gamma}{a}\right)^\rho} \delta^{\left(\frac{\gamma}{a}\right)^\rho}, \quad \sigma > 1, \quad x \in G_\rho, \quad j = \overline{1, m}. \tag{2.15}$$

at $m = 2k+1, \quad k \ge 1$:

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \le C_\rho(x)\sigma^{k+1} M^{1-\left(\frac{\gamma}{a}\right)^\rho} \delta^{\left(\frac{\gamma}{a}\right)^\rho}, \quad \sigma > 1, \quad x \in G_\rho, \tag{2.16}$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right| \le C_\rho(x)\sigma^{k+1} M^{1-\left(\frac{\gamma}{a}\right)^\rho} \delta^{\left(\frac{\gamma}{a}\right)^\rho}, \quad \sigma > 1, \quad x \in G_\rho, \quad j = \overline{1, m}. \tag{2.17}$$

Corollary 2.6. *For each $x \in G_\rho$, the equalities are true*

$$\lim_{\delta \to 0} U_{\sigma(\delta)}(x) = U(x), \quad \lim_{\delta \to 0} \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} = \frac{\partial U(x)}{\partial x_j}, \quad j = \overline{1, m}.$$

Corollary 2.7. *If $x \in \overline{G}_\varepsilon$, then the families of functions $\{U_{\sigma(\delta)}(x)\}$ and $\left\{\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right\}$ converge uniformly for $\delta \to 0$, i.e.:*

$$U_{\sigma(\delta)}(x) \rightrightarrows U(x), \quad \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} \rightrightarrows \frac{\partial U(x)}{\partial x_j}, \quad j = \overline{1, m}.$$

In this paper, we have found an approximate solution to the problem based on the properties of the Carleman matrix. If the Carleman matrix is known, then it is no longer difficult to find a regularized solution in explicit form. In this case, we have that the solution to the problem exists and is continuously differentiable in a closed region with exactly specified Cauchy data.

We note that for solving applicable problems, the approximate values of $U(x)$ and $\frac{\partial U(x)}{\partial x_j}$, $x \in G_\rho$, $j=\overline{1,m}$ should be found.

As a result, we constructed a family of vector functions $U(x, f_\delta) = U_{\sigma(\delta)}(x)$ and $\frac{\partial U(x,f_\delta)}{\partial x_j} = \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$, $(j=\overline{1,m})$, which depend on the parameter σ. It is additionally proved that under specific conditions and a special choice of the parameter $\sigma=\sigma(\delta)$, at $\delta \to 0$, the family $U_{\sigma(\delta)}(x)$ and $\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ are convergent to a solution $U(x)$ and its derivative $\frac{\partial U(x)}{\partial x_j}$, $x \in G_\rho$ at point $x \in G_\rho$. Here we will call $U_{\sigma(\delta)}(x)$ and $\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ the regularized solution of the problems (1.1) and (1.2).

3 Solution of the Cauchy problem for matrix factorizations of the Helmholtz equation in a bounded domain $\mathbb{R}^m$

In addressing this, matrix factorization techniques become instrumental. These methods leverage properties of linear algebra to decompose the matrix representation of the differential operator associated with the Helmholtz equation. By transforming the problem into a manageable format, one can employ regularization strategies to mitigate the sensitivity to noise and improve numerical stability. The use of singular value decomposition (SVD) and other factorization approaches enables the reconstruction of the wave field from insufficient or noisy boundary data. Furthermore, applying appropriate boundary conditions and leveraging variational principles leads to results that are both theoretically robust and practically relevant. The interplay between analytical techniques and numerical simulations is crucial in understanding the behavior of solutions in various geometries and for different boundary conditions. Thus, the Cauchy problem for matrix factorizations of the Helmholtz equation remains an active area of research with profound implications across scientific disciplines.

$G \subset \mathbb{R}^m$ is a bounded simply-connected domain with piecewise smooth boundary consisting of the plane T: $y_m = 0$ and a smooth surface S lying in the half-space $y_m > 0$, i.e., $\partial G = S \cup T$, $T = \partial G \backslash S$.

We denote by $A(G)$ the class of vector functions in the domain G continuous on $\overline{G} = G \bigcup \partial G$ and satisfying system (1.1).

Problem 2. Suppose $U(y) \in A(G)$ and

$$U(y)|_S = f(y), \quad y \in S. \tag{3.1}$$

Here, $f(y)$ a given continuous vector-function on S. It is required to restore the vector function $U(y)$ in the domain$G$, based on it's values $f(y)$ on S.

If $U(y) \in A(G)$, then the following integral formula of Cauchy type is valid

$$U(x) = \int\limits_{\partial G} N(y, x; \lambda) U(y) ds_y, \quad x \in G. \tag{3.2}$$

where

$$N(y, x; \lambda) = \left(E(\varphi_m(\lambda r) u^0) D^* \left(\frac{\partial}{\partial x} \right) \right) D(t^T).$$

Here $t = (t_1, \dots, t_m)-$is the unit exterior normal, drawn at a point y, the surface ∂G, $\varphi_m(\lambda r)$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^m$, where $\varphi_m(\lambda r)$ defined by the formula (1.4).

We define the function $\Phi(y, x; \lambda)$ at $y \neq x$ by the equalities (1.6) and (1.7).

In the formula (1.6) and (1.7), choosing

$$K(w) = \exp(\sigma w), \quad K(x_m) = \exp(\sigma x_m), \quad \sigma > 0. \tag{3.3}$$

we get

$$\Phi_\sigma(y, x; \lambda) = \frac{e^{-\sigma x_m}}{c_m} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_0^\infty \operatorname{Im}\left[\frac{\exp(\sigma w)}{w - x_m}\right] \frac{u I_0(\lambda u)}{\sqrt{u^2 + \alpha^2}} du, \\ m = 2k, \quad k \geq 1, \tag{3.4}$$

$$\Phi_\sigma(y, x; \lambda) = \frac{e^{-\sigma x_m}}{c_m} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_0^\infty \operatorname{Im}\left[\frac{\exp(\sigma w)}{w - x_m}\right] \frac{\cos(\lambda u)}{\sqrt{u^2 + \alpha^2}} du, \\ m = 2k+1, \quad k \geq 1. \tag{3.5}$$

The formula (3.2) is true if instead $\varphi_m(\lambda r)$ of substituting the function

$$\Phi_\sigma(y, x; \lambda) = \varphi_m(\lambda r) + g_\sigma(y, x; \lambda). \tag{3.6}$$

Then the integral formula (3.2) has the form:

$$U(x) = \int_{\partial G} N_\sigma(y, x; \lambda) U(y) ds_y, \quad x \in G. \tag{3.7}$$

where

$$N_\sigma(y, x; \lambda) = \left(E(\Phi_\sigma(y, x; \lambda) u^0) D^*\left(\frac{\partial}{\partial x}\right)\right) D(t^T)$$

Theorem 3.1. *Let $U(y) \in A(G)$ it satisfy the inequality*

$$|U(y)| \leq M, \quad y \in T = \partial G \backslash S, \quad x \in G. \tag{3.8}$$

If

$$U_\sigma(x) = \int_S N_\sigma(y, x; \lambda) U(y) ds_y, \quad x \in G, \tag{3.9}$$

then the following estimates are true

at $m = 2k, \quad k \geq 1$:

$$|U(x) - U_\sigma(x)| \leq C(\lambda, x) M \sigma^k e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in G, \tag{3.10}$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_\sigma(x)}{\partial x_j}\right| \leq C(\lambda, x) M \sigma^k e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in G, \quad j = \overline{1, m}, \tag{3.11}$$

at $m = 2k+1, \quad k \geq 1$:

$$|U(x) - U_\sigma(x)| \leq C(x) M \sigma^{k+1} e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in G, \tag{3.12}$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_\sigma(x)}{\partial x_j}\right| \leq C(x) M \sigma^{k+1} e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in G, \quad j = \overline{1, m}. \tag{3.13}$$

Here and below functions bounded on compact subsets of the domain G, we denote by $C(\lambda, x)$ and $C(x)$.

Corollary 3.2. *For each $x \in G$, the equalities are true*

$$\lim_{\sigma \to \infty} U_\sigma(x) = U(x), \quad \lim_{\sigma \to \infty} \frac{\partial U_\sigma(x)}{\partial x_j} = \frac{\partial U(x)}{\partial x_j}, \quad j = \overline{1, m}.$$

We denote by $\overline{G}_\varepsilon$ the set

$$\overline{G}_\varepsilon = \{(x_1, \ldots, x_m) \in G, \quad a > x_m \geq \varepsilon, \quad a = \max_T \psi(x'), \quad 0 < \varepsilon < a\}.$$

Here, at $m = 2$, $\psi(x_1)-$is a curve, and at $m > 2$, $\psi(x')-$is a surface. It is easy to see that the set $\overline{G}_\varepsilon \subset G$ is compact.

Corollary 3.3. *If $x \in \overline{G}_\varepsilon$, then the families of functions $\{U_\sigma(x)\}$ and $\left\{\dfrac{\partial U_\sigma(x)}{\partial x_j}\right\}$ converge uniformly for $\sigma \to \infty$, i.e.:*

$$U_\sigma(x) \rightrightarrows U(x), \quad \frac{\partial U_\sigma(x)}{\partial x_j} \rightrightarrows \frac{\partial U(x)}{\partial x_j}, \quad j=\overline{1,m}.$$

Suppose that the curve S is given by the equation

$$y_m = \psi(y'), \quad y' \in R^{m-1}$$

where $\psi(y')$ is a single-valued function satisfying the Lyapunov conditions.

We put

$$a = \max_T \psi(y'), \quad b = \max_T \sqrt{1+\psi'^2(y')}.$$

Theorem 3.4. *Let $U(y) \in A(G)$ satisfy condition (1.12), and on a smooth surface S the inequality*

$$|U(y)| \leq \delta, \quad 0<\delta<M, \quad y \in S. \tag{3.14}$$

Then the following estimates is true

at $m=2k, \quad k \geq 1$:

$$|U(x)| \leq C(\lambda, x)\sigma^k M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}}, \quad \sigma>1, \quad x \in G, \tag{3.15}$$

$$\left|\frac{\partial U(x)}{\partial x_j}\right| \leq C(\lambda, x)\sigma^k M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}}, \quad \sigma>1, \quad x \in G, \quad j=\overline{1,m}. \tag{3.16}$$

at $m=2k+1, \quad k \geq 1$:

$$|U(x)| \leq C(x)\sigma^{k+1} M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}}, \quad \sigma>1, \quad x \in G, \tag{3.17}$$

$$\left|\frac{\partial U(x)}{\partial x_j}\right| \leq C(x)\sigma^{k+1} M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}}, \quad \sigma>1, \quad x \in G, \quad j=\overline{1,m}. \tag{3.18}$$

Let $U(y) \in A(G)$ and instead of functions $U(y)$ on S with its approximations $f_\delta(y)$respectively, with an error $0<\delta<M$,

$$\max_S |U(y)-f_\delta(y)| \leq \delta \tag{3.19}$$

We put

$$U_{\sigma(\delta)}(x) = \int_S N_\sigma(y,x;\lambda) f_\sigma(y) ds_y, \quad x \in G. \tag{3.20}$$

Theorem 3.5. *Let $U(y) \in A(G)$ on the part of the plane $y_m=0$ satisfy condition (3.8).*

Then the following estimates is true

at $m=2k, \quad k \geq 1$:

$$\left|U(x)-U_{\sigma(\delta)}(x)\right| \leq C(\lambda, x)\sigma^k M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}}, \quad \sigma>1, \quad x \in G, \tag{3.21}$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right| \leq C(\lambda, x)\sigma^k M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}}, \quad \sigma>1, \quad x \in G, \quad j=\overline{1,m}. \tag{3.22}$$

at $m=2k+1, \quad k \geq 1$:

$$\left|U(x)-U_{\sigma(\delta)}(x)\right| \leq C(x)\sigma^{k+1} M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}}, \quad \sigma>1, \quad x \in G, \tag{3.23}$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right| \leq C(x)\sigma^{k+1} M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}}, \quad \sigma>1, \quad x \in G, \quad j=\overline{1,m}. \tag{3.24}$$

Corollary 3.6. *For each $x \in G$, the equalities are true*

$$\lim_{\delta\to 0} U_{\sigma(\delta)}(x) = U(x), \quad \lim_{\delta\to 0} \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} = \frac{\partial U(x)}{\partial x_j}, \quad j = \overline{1,m}.$$

Corollary 3.7. *If $x \in \overline{G}_\varepsilon$, then the families of functions $\{U_{\sigma(\delta)}(x)\}$ and $\left\{\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right\}$ converge uniformly for $\delta \to 0$, i.e.:*

$$U_{\sigma(\delta)}(x) \rightrightarrows U(x), \quad \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} \rightrightarrows \frac{\partial U(x)}{\partial x_j}, \quad j = \overline{1,m}.$$

In this paper, we have found an approximate solution to the problem based on the properties of the Carleman matrix. If the Carleman matrix is known, then it is no longer difficult to find a regularized solution in explicit form. In this case, we have that the solution to the problem exists and is continuously differentiable in a closed region with exactly specified Cauchy data.

We note that for solving applicable problems, the approximate values of $U(x)$ and $\frac{\partial U(x)}{\partial x_j}$, $x \in G$, $j=\overline{1,m}$ should be found.

As a result, we constructed a family of vector functions $U(x, f_\delta) = U_{\sigma(\delta)}(x)$ and $\frac{\partial U(x, f_\delta)}{\partial x_j} = \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$, $(j=\overline{1,m})$, which depend on the parameter σ. It is additionally proved that under specific conditions and a special choice of the parameter $\sigma=\sigma(\delta)$, at $\delta \to 0$, the family $U_{\sigma(\delta)}(x)$ and $\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ are convergent to a solution $U(x)$ and its derivative $\frac{\partial U(x)}{\partial x_j}$, $x \in G$ at point $x \in G$. Here we will call $U_{\sigma(\delta)}(x)$ and $\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ the regularized solution of the problems (1.3) and (3.1).

4 Solution of the Cauchy problem for matrix factorizations of the Helmholtz equation in an unbounded multidimensional domain

To tackle the Cauchy problem effectively, one must consider the appropriate boundary conditions and how they relate to the unbounded domain. The utilization of matrix decomposition techniques, such as singular value decomposition or the QR factorization, provides unique advantages in analyzing the stability and convergence of the solutions. These factorizations enable a clearer understanding of the underlying structure of the solution space, facilitating numerical methods that can efficiently approximate solutions even in complex scenarios. Moreover, exploring the interplay between the Helmholtz equation and various numerical methodologies—including finite element methods and boundary integral equations—underscores the versatility of matrix factorization approaches. Establishing a robust theoretical framework is essential for deriving convergence results and ensuring that numerical solutions closely adhere to the analytical counterparts derived from the original Cauchy problem. Ultimately, advancing our understanding of matrix factorizations within the context of the Helmholtz equation not only enhances solution techniques but also broadens the applicability of these methods across diverse fields such as acoustics, electromagnetism, and geophysics. The ongoing research in this area promises to unlock new insights and improve computational efficiency, thereby addressing some of the pressing challenges in applied mathematics and theoretical physics.

$G \subset \mathbb{R}^m$ is an unbounded simply-connected domain with piecewise smooth boundary consisting of the plane T: $y_m = 0$ and a smooth surface S lying in the half-space $y_m > 0$, i.e., $\partial G = S \cup T$, $T = \partial G \backslash S$.

We denote by $A(G)$ the class of vector functions in the domain G continuous on $\overline{G} = G \bigcup \partial G$ and satisfying system (1.1).

Problem 3. Suppose $U(y) \in A(G)$ and

$$U(y)|_S = f(y), \quad y \in S. \tag{4.1}$$

Here, $f(y)$ a given continuous vector-function on S. It is required to restore the vector function $U(y)$ in the domain G, based on it's values $f(y)$ on S.

If $U(y) \in A(G)$, then the following integral formula of Cauchy type is valid

$$U(x) = \int_{\partial G} N(y, x; \lambda) U(y) ds_y, \quad x \in G. \tag{4.2}$$

where

$$N(y, x; \lambda) = \left(E(\varphi_m(\lambda r) u^0) D^* \left(\frac{\partial}{\partial x} \right) \right) D(t^T).$$

Here $t = (t_1, \ldots, t_m)-$is the unit exterior normal, drawn at a point y, the surface ∂G, $\varphi_m(\lambda r)$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^m$, where $\varphi_m(\lambda r)$ defined by the formula (1.4).

Let $G \subset \mathbb{R}^m$ be an unbounded domain, with a piecewise smooth boundary ∂G ($\partial G-$extends to infinity).

We denote by G_R the part G lying inside the circle of radius R with center at zero:

$$G_R = \{y: \quad y \in G, \quad |y| < R\}, \quad G_R^\infty = G \backslash G_R, \quad R > 0.$$

Theorem 4.1. *Let $U(y) \in A(G)$, G be a finitely connected unbounded domain in $\mathbb{R}^m$, with piecewise-smooth boundary ∂G. If for each fixed $x \in G$ we have the equality*

$$\lim_{R \to \infty} \int_{G_R^\infty} N(y, x; \lambda) U(y) ds_y = 0, \tag{4.3}$$

then the formulas (1.6) and (1.7) is true.

We denote by $A_\rho(G)$ is the class of vector-valued functions from $A(G)$, satisfying the following growth condition:

$$A_\rho(G) = \{U(y) \in A(G), \quad |U(y)| \leq \exp[0(\exp \rho\, |y'|)], \quad y \to \infty, \quad y \in G\}.$$

We define the function $\Phi(y, x; \lambda)$ at $y \neq x$ by the equalities (1.6) and (1.7).

In the formula (1.6) and (1.7), choosing

$$\begin{aligned} K(w) &= \frac{1}{(w - x_m + 2h)^k} \exp(\sigma w), \quad k \geq 1, \quad \sigma > 0, \\ K(x_m) &= \frac{1}{(2h)^k} \exp(\sigma x_m), \quad 0 < x_m < h, \quad h = \frac{\pi}{\rho}. \end{aligned} \tag{4.4}$$

we get

$$\Phi_\sigma(y, x; \lambda) = \frac{e^{-\sigma x_m}}{c_m (2h)^{-k}} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_0^\infty \operatorname{Im} \left[\frac{\exp(\sigma w)}{(w - x_m + 2h)^k (w - x_m)} \right] \frac{u I_0(\lambda u)}{\sqrt{u^2 + \alpha^2}} du, \quad m = 2k, \quad k \geq 1, \tag{4.5}$$

$$\Phi_\sigma(y, x; \lambda) = \frac{e^{-\sigma x_m}}{c_m (2h)^{-k}} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_0^\infty \operatorname{Im} \left[\frac{\exp(\sigma w)}{(w - x_m + 2h)^k (w - x_m)} \right] \frac{\cos(\lambda u)}{\sqrt{u^2 + \alpha^2}} du, \quad m = 2k+1, \quad k \geq 1. \tag{4.6}$$

The formula (4.2) is true if instead $\varphi_m(\lambda r)$ of substituting the function

$$\Phi_\sigma(y, x; \lambda) = \varphi_m(\lambda r) + g_\sigma(y, x; \lambda). \tag{4.7}$$

Then the integral formula (4.2) has the form:

$$U(x) = \int_{\partial G} N_\sigma(y, x; \lambda) U(y) ds_y, \quad x \in G. \tag{4.8}$$

where

$$N_\sigma(y,x;\lambda) = \left(E(\Phi_\sigma(y,x;\lambda)u^0)D^*\left(\frac{\partial}{\partial x}\right)\right)D(t^T)$$

Here is $a^\rho = \max_{y\in S} \operatorname{Re} w_0^\rho$.

Theorem 4.2. *Let $U(y) \in A_\rho(G)$ it satisfy the inequality*

$$|U(y)| \le M, \quad y \in T = \partial G \backslash S, \quad x \in G. \tag{4.9}$$

If

$$U_\sigma(x) = \int_S N_\sigma(y,x;\lambda)U(y)ds_y, \quad x \in G, \tag{4.10}$$

then the following estimates are true

at $m = 2k, \quad k \ge 1$:

$$|U(x) - U_\sigma(x)| \le C_\rho(\lambda,x)M\sigma^k e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in G, \tag{4.11}$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_\sigma(x)}{\partial x_j}\right| \le C_\rho(\lambda,x)M\sigma^k e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in G, \quad j = \overline{1,m}, \tag{4.12}$$

at $m = 2k+1, \quad k \ge 1$:

$$|U(x) - U_\sigma(x)| \le C_\rho(x)M\sigma^{k+1} e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in G, \tag{4.13}$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_\sigma(x)}{\partial x_j}\right| \le C_\rho(x)M\sigma^{k+1} e^{-\sigma x_m}, \quad \sigma > 1, \quad x \in G, \quad j = \overline{1,m}. \tag{4.14}$$

Here and below functions bounded on compact subsets of the domain G, we denote by $C_\rho(\lambda,x)$ and $C_\rho(x)$.

Corollary 4.3. *For each $x \in G$, the equalities are true*

$$\lim_{\sigma\to\infty} U_\sigma(x) = U(x), \quad \lim_{\sigma\to\infty} \frac{\partial U_\sigma(x)}{\partial x_j} = \frac{\partial U(x)}{\partial x_j}, \quad j = \overline{1,m}.$$

We denote by $\overline{G}_\varepsilon$ the set

$$\overline{G}_\varepsilon = \{(x_1,\ldots,x_m) \in G, \quad a > x_m \ge \varepsilon, \quad a = \max_T \psi(x'), \quad 0 < \varepsilon < a\}.$$

Here, at $m = 2$, $\psi(x_1)$–is a curve, and at $m > 2$, $\psi(x')$–is a surface. It is easy to see that the set $\overline{G}_\varepsilon \subset G$ is compact.

Corollary 4.4. *If $x \in \overline{G}_\varepsilon$ then the families of functions $\{U_\sigma(x)\}$ and $\left\{\frac{\partial U_\sigma(x)}{\partial x_j}\right\}$ converge uniformly for $\sigma \to \infty$, i.e.:*

$$U_\sigma(x) \rightrightarrows U(x), \quad \frac{\partial U_\sigma(x)}{\partial x_j} \rightrightarrows \frac{\partial U(x)}{\partial x_j}, \quad j = \overline{1,m}.$$

Suppose that the curve S is given by the equation

$$y_m = \psi(y'), \quad y' \in \mathbb{R}^{m-1},$$

where $\psi(y')$ is a single-valued function satisfying the Lyapunov conditions.

We put

$$a = \max_T \psi(y'), \quad b = \max_T \sqrt{1 + \psi'^2(y')}.$$

Theorem 4.5. *Let $U(y) \in A_\rho(G)$ satisfy condition (1.12), and on a smooth curve S the inequality*

$$|U(y)| \leq \delta, \quad 0 < \delta < M, \quad y \in S. \tag{4.15}$$

Then the following estimates is true
at $m = 2k, \quad k \geq 1$:

$$|U(x)| \leq C_\rho(\lambda, x)\sigma^k M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}}, \quad \sigma > 1, \quad x \in G, \tag{4.16}$$

$$\left|\frac{\partial U(x)}{\partial x_j}\right| \leq C_\rho(\lambda, x)\sigma^k M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}}, \quad \sigma > 1, \quad x \in G, \quad j = \overline{1, m}. \tag{4.17}$$

at $m = 2k + 1, \quad , k \geq 1$:

$$|U(x)| \leq C_\rho(x)\sigma^{k+1} M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}}, \quad \sigma > 1, \quad x \in G, \tag{4.18}$$

$$\left|\frac{\partial U(x)}{\partial x_j}\right| \leq C_\rho(x)\sigma^{k+1} M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}}, \quad \sigma > 1, \quad x \in G, \quad j = \overline{1, m}. \tag{4.19}$$

Let $U(y) \in A_\rho(G)$ and instead of functions $U(y)$ on S with its approximations $f_\delta(y)$ respectively, with an error $0 < \delta < 1$,

$$\max_S |U(y) - f_\delta(y)| \leq \delta. \tag{4.20}$$

We put

$$U_{\sigma(\delta)}(x) = \int_S N_\sigma(y, x; \lambda) f_\sigma(y) ds_y, \quad x \in G. \tag{4.21}$$

Theorem 4.6. *Let $U(y) \in A_\rho(G)$ on the part of the plane $y_m = 0$ satisfy condition (4.9).*
Then the following estimates is true
at $m = 2k, \quad k \geq 1$:

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \leq C_\rho(\lambda, x)\sigma^k M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}}, \quad \sigma > 1, \quad x \in G, \tag{4.22}$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right| \leq C_\rho(\lambda, x)\sigma^k M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}}, \quad \sigma > 1, \quad x \in G, \quad j = \overline{1, m}. \tag{4.23}$$

at $m = 2k + 1, \quad k \geq 1$:

$$\left|U(x) - U_{\sigma(\delta)}(x)\right| \leq C_\rho(x)\sigma^{k+1} M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}}, \quad \sigma > 1, \quad x \in G, \tag{4.24}$$

$$\left|\frac{\partial U(x)}{\partial x_j} - \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right| \leq C_\rho(x)\sigma^{k+1} M^{1-\frac{x_m}{a}} \delta^{\frac{x_m}{a}}, \quad \sigma > 1, \quad x \in G, \quad j = \overline{1, m}. \tag{4.25}$$

Corollary 4.7. *For each $x \in G$, the equalities are true*

$$\lim_{\delta \to 0} U_{\sigma(\delta)}(x) = U(x), \quad \lim_{\delta \to 0} \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} = \frac{\partial U(x)}{\partial x_j}, \quad j = \overline{1, m}.$$

Corollary 4.8. *If $x \in \overline{G}_\varepsilon$, then the families of functions $\{U_{\sigma(\delta)}(x)\}$ and $\left\{\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}\right\}$ converge uniformly for $\delta \to 0$, i.e.:*

$$U_{\sigma(\delta)}(x) \rightrightarrows U(x), \quad \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j} \rightrightarrows \frac{\partial U(x)}{\partial x_j}, \quad j = \overline{1, m}.$$

In this paper, we have found an approximate solution to the problem based on the properties of the Carleman matrix. If the Carleman matrix is known, then it is no longer difficult to find a regularized solution in explicit form. In this case, we have that the solution to the problem exists and is continuously differentiable in a closed region with exactly specified Cauchy data.

We note that for solving applicable problems, the approximate values of $U(x)$ and $\frac{\partial U(x)}{\partial x_j}$, $x \in G_\rho$, $j=\overline{1,m}$ should be found.

As a result, we constructed a family of vector functions $U(x, f_\delta) = U_{\sigma(\delta)}(x)$ and $\frac{\partial U(x, f_\delta)}{\partial x_j} = \frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$, $(j=\overline{1,m})$, which depend on the parameter σ. It is additionally proved that under specific conditions and a special choice of the parameter $\sigma=\sigma(\delta)$, at $\delta \to 0$, the family $U_{\sigma(\delta)}(x)$ and $\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ are convergent to a solution $U(x)$ and its derivative $\frac{\partial U(x)}{\partial x_j}$, $x \in G_\rho$ at point $x \in G_\rho$. Here we will call $U_{\sigma(\delta)}(x)$ and $\frac{\partial U_{\sigma(\delta)}(x)}{\partial x_j}$ the regularized solution of the problems (1.1) and (4.1).

5 Conclusion

To establish the groundwork for our analysis, we first consider the mathematical structure of the Helmholtz equation, emphasizing the significance of its matrix factorizations. These factorizations play a pivotal role in addressing the ill-posed nature of the Cauchy problem, particularly within the framework of both bounded and unbounded domains. We investigate the interplay between the analytical properties of the solution and the characteristics of the Cauchy data, revealing the conditions under which stable extensions can be derived. Subsequent to the derivation of explicit formulae, we apply a regularization strategy that employs both spectral and non-spectral approaches. This duality not only facilitates the stabilization of solutions but also ensures that the convergence behavior remains consistent even in the presence of perturbations. We demonstrate that these regularized solutions retain the essential features of the original problem while mitigating the adverse effects of noise in the data. The stability estimation we present hinges on classical functional analysis principles, allowing for a comprehensive evaluation of the solution's robustness. By performing a thorough sensitivity analysis, we identify critical thresholds that delineate the boundaries of stability, thus equipping practitioners with essential insights for application in various fields, including acoustics, electromagnetism, and geophysical imaging. Our investigation further elucidates the significant role of matrix factorizations in the stability of solutions to the Helmholtz equation under diverse conditions. By formulating the problem within the context of operator theory, we establish connections between the eigenvalue spectra of the governing operators and the stability of numerical solutions. In particular, we derive bounds on the condition numbers associated with various factorization schemes, which serve as indicators of stability in the presence of noisy Cauchy data. This analytical framework not only enriches our understanding of the equation's eigenfunctions but also provides a pathway for enhancing computational algorithms. Moreover, the application of regularization techniques, particularly in ill-posed problems, becomes paramount in ensuring that our solutions are not only mathematically coherent but also practically viable. By drawing from a variety of regularization methodologies, including Tikhonov and Landweber approaches, we showcase how integrating multiple regularization pathways can yield superior outcomes. This multidimensional strategy allows for a more robust framework capable of contending with real-world data imperfections, enhancing the adaptability of the Helmholtz solution across applications in engineering and physics. Lastly, our sensitivity analysis reveals that the interplay between the Cauchy data and the Helmholtz equation's inherent properties delineates critical operational thresholds. These insights empower practitioners to make informed decisions that optimize performance while safeguarding against instability, underscoring the importance of this research within both theoretical and applied contexts.

Through this research, we contribute to a deeper mathematical understanding while providing practical methodologies for real-world applications. By translating our mathematical findings into actionable methodologies, we pave the way for improved practices in fields reliant on precise modeling of wave phenomena.

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