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Coercive Estimates and Separability for the Fourth-Order Operator in Weight Space

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Abstract

The paper investigates the coercive properties of a fourth-order operator in the weight space $L_{2,\rho}(R^n)$. This paper presents a detailed investigation into the coercive properties and separability conditions of a fourth-order differential operator within a weighted Hilbert space framework. The motivation stems from the critical role such operators play in mathematical physics and engineering, particularly in modeling elastic structures and complex boundary value problems. While the theory of second-order and biharmonic operators is well developed, there remains a need to deepen the understanding of higher-order operators, especially in non-uniform or weighted settings. We begin by defining the analytical setting, including the class of admissible weight functions and functional spaces, and formulate the problem through a general fourth-order elliptic equation. Building upon existing approaches in the theory of coercivity and separability, we derive new sufficient conditions under which the operator admits a coercive inequality. These inequalities are essential in demonstrating well-posedness and are shown to lead directly to separability, which ensures the operator's spectral decomposition and solution uniqueness. The core result of the paper is a theorem establishing that, under certain regularity and boundedness conditions on the weight functions, the considered operator is separable in the weighted space. The proof employs a combination of integration by parts, approximation techniques, and delicate functional estimates that rely on properties of the weight function and the structure of the differential operator. This work generalizes prior results on coercive solvability from second-order settings to fourth-order frameworks and introduces a systematic method to verify separability using inequalities derived from the operator's coercive structure. As such, the results have broader implications for the study of nonlinear and non-divergent elliptic operators in weighted domains.

Keywords: Separability, Coercive estimates, Fourth-order differential operator, Weight space, Elliptic equations.

1 | Introduction

The study of differential operators in functional spaces plays a vital role in modern analysis, particularly in the theory of Partial Differential Equations (PDEs) and mathematical physics. Among the various types of differential operators, those of higher order, such as fourth-order operators, arise naturally in a broad spectrum of physical models, including elasticity theory, plate and beam problems, quantum mechanics, and other applied disciplines. Understanding the analytic and spectral properties of these operators, especially in weighted spaces, remains a subject of considerable theoretical and practical interest. One key concept in this

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area is separability. This structural property ensures that a differential operator behaves well under decomposition and allows solutions to be obtained via expansion in terms of suitable basis functions. Separability has direct implications for solvability, uniqueness, and numerical approximation of differential equations. The term "separability" in the context of differential operators was introduced and developed by Everitt and Giertz [1–4], who studied the Sturm–Liouville operator and established foundational results. Later developments, particularly those by Boymatov [2] and Otelbaev [5], and their followers, extended these ideas to broader classes of elliptic and higher-order operators. Closely related to separability is the concept of coercivity, which is a type of a priori estimate that provides lower bounds for operator norms in terms of function norms. Coercive estimates are essential tools in proving well-posedness of boundary value problems and in establishing energy estimates for PDEs. The work of Otelbaev [5] and others has shown that coercivity conditions often serve as sufficient criteria for separability. However, the exact relationship between these two concepts becomes more intricate in weighted function spaces and for higher-order operators. While much of the classical theory focused on second-order elliptic operators, recent research efforts have turned toward the more challenging fourth-order operators. These operators, due to their increased complexity and the presence of higher derivatives, demand more refined analytical techniques, especially when considered in weighted spaces where the geometry and growth conditions of the weight functions significantly influence the analysis. In this paper, we examine a fourth-order differential operator acting in a weighted Hilbert space. We aim to derive new coercive inequalities that serve as sufficient conditions for separability. By precisely characterizing the classes of admissible weight functions and functions involved in the operator, we formulate a theorem that guarantees separability under specific analytic assumptions. The main novelty of our work lies in extending existing results for second-order and biharmonic operators to more general fourth-order operators, and doing so within the broader framework of weighted spaces, which are often encountered in realistic models where non-uniformities or singularities are present.

The remainder of the article is structured as follows. In Section 2, we provide the precise formulation of the problem and state the main theorem. Section 3 develops the auxiliary lemmas and technical estimates needed for the proof. Section 4 is devoted to proving the main result, including detailed coercive inequalities and the role they play in establishing separability. We conclude by summarizing the implications of our findings and suggesting directions for further research in the analysis of nonlinear and higher-order differential operators in weighted functional frameworks. They examined the separability of the Sturm–Liouville operator and its degrees. A significant contribution to the future development of the theory of separability of differential expressions was introduced by Boimatov [2], Otelbaev [5], and their disciples. The separability of differential expressions and the corresponding coercivity inequalities have been studied in many papers (See [1], [2], [5], [6] and references therein). Coercivity estimates and separability for second-order elliptic differential equations were studied in [3]. In [4] and [7] investigated the separability of a biharmonic operator with a matrix potential. The paper [7] is devoted to the study of nonlinear elliptic differential operators of nondivergent form in a weighted space. In this paper, we consider an operator of the fourth order in the weight space $L_{2,\rho}(R^n)$, establish the corresponding coercivity inequalities and, on the basis of these inequalities, obtain new sufficient conditions of separability for this operator.

2 | The Formulation of the Main Result of This Work

Let's introduce a weight $L_{2,\rho}(R^n)$ with finite

$$\| u, L_{2,\rho}(R^n) \| = \left\{ \int_{R^n} \rho(x) |u(x)|^2 dx \right\}^{\frac{1}{2}},$$

where ϱ is a positive function.

The space $L_{2,\rho}(R^n)$ is a Hilbert space, and in it the scalar product is defined by the equality

$$\langle u, \upsilon\rangle_\rho = \int_{R^n} \rho(x)u(x)\overline{\upsilon(x)}dx.$$

In the space $L_{2,\rho}(R^n)$ consider the differential equation:

$$L[u] = \sum_{i=1}^{n} \frac{\partial^4 u(x)}{\partial x_i^4} + V(x) \cdot u(x) = f(x), \tag{1}$$

where $V(x)$- is a positive function.

Definition 1. *Eq. (1)* and the corresponding differential-operator are called separable in the $L_{2,\rho}(R^n)$, если $\frac{\partial^4 u(x)}{\partial x_i^4}, V(x) \cdot u(x) \in L_{2,\rho}(R^n)$ для всех $u(x) \in L_{2,\rho}(R^n) \cap W_{2,loc}^4(R^n)$ such that $f(x) \in L_{2,\rho}(R^n)$.

Definition 2. We say that the function belongs to a class if the following conditions are satisfied for all:

$$\sum_{i=1}^{n} \left\| V^{-\frac{1}{2}}(x) \frac{\partial^2 V(x)}{\partial x_i^2} V^{-1}(x) \right\| \le \sigma_1. \tag{2}$$

$$\left\| V^{-\frac{1}{2}}(x) \frac{\partial V(x)}{\partial x_i} \frac{\partial u}{\partial x_i} \right\| \le \sigma_2 \left\| V^{\frac{1}{2}}(x) \frac{\partial^2 u}{\partial x_i^2} \right\|. \tag{3}$$

Let us formulate the main result of the paper. The following is true.

Theorem 1. Let the function $V(x)$ belong to the class $T_{\sigma_1,\sigma_2}^{\delta_1,\delta_2}$. Let the weight function $\rho(x)$ belong to the class $C^2(R^n)$, and for all $x \in R^n$ inequalities hold:

$$\left\| \rho^{-1} \frac{\partial \rho}{\partial x_i} V^{-\frac{1}{2}} \frac{\partial u}{\partial x_i} \right\| \le \sigma_3 \parallel V^{\frac{1}{2}} \frac{\partial^2 u}{\partial x_i^2} |. \tag{4}$$

$$\sum_{i=1}^{n} \left\| \rho^{-1} \frac{\partial \rho}{\partial x_i} V^{-\frac{1}{2}} \frac{\partial V(x)}{\partial x_i} V^{-1} \right\| \le \sigma_4. \tag{5}$$

$$\sum_{i=1}^{n} \left\| \rho^{-1} \frac{\partial^2 \rho}{\partial x_i^2} V^{-\frac{1}{2}} \right\| \le \delta_1. \tag{6}$$

$$\left\| \rho^{-1} \frac{\partial \rho}{\partial x_i} \frac{\partial u}{\partial x_i} \right\| \le \delta_2 \left| \frac{\partial^2 u}{\partial x_i^2} \right|. \tag{7}$$

Then, if the conditions specified in *Eq. (8)* are satisfied.

$$\delta_1^2 < 4(1 - 2\delta_2), \delta_1 + \sigma_1 + 2\sigma_4 < 2\sqrt{1 - 2\sigma_2 - 2\sigma_3}. \tag{8}$$

Eq. (1) is separable in the space $L_{2,\rho}(R^n)$, and for all solutions of $u(x) \in L_{2,\rho}(R^n) \cap W_{2,loc}^4(R^n)$ equations $\sum_{i=1}^{n} \frac{\partial^4 u(x)}{\partial x_i^4} + V(x)u(x) = f(x)$.

With the right part $f(x) \in L_{2,\rho}(R^n)$ the following coercive inequality is satisfied:

$$\parallel \sum_{i=1}^{n} \frac{\partial^4 u(x)}{\partial x_i^4}, L_{2,\rho}(R^n) \parallel + \parallel V(x)u(x), L_{2,\rho}(R^n) + \sum_{i=1}^{n} \parallel V^{\frac{1}{2}}(x) \frac{\partial u^2(x)}{\partial x_i^2}, L_{2,\rho}(R^n) \parallel \le M \parallel f(x), L_{2,\rho}(R^n), \tag{9}$$

where the positive number M is independent of $u(x)$, $f(x)$.

3 | Auxiliary Lemmas

We will frequently use the following lemma, which is proved by integration by parts.

Lemma 1. Then, for any two functions φ and ψ, the following equality holds:

$$\langle \omega_1', \psi\omega_2\rangle = -\langle \omega_1, \psi'\omega_2\rangle - \langle \omega_1, \psi\omega_2'\rangle, (\omega_1, \omega_2 \in W_{2,loc}^1(R), \psi \in C_0^\infty(R^n)). \quad \textbf{(10)}$$

Lemma 2. Let the function $f(x)$ in *Eq. (1)* belong to the space $L_{2,\rho}(R^n)$, the function $u(x)$ belong to the space $L_{2,\rho}(R^n) \cap W_{2,loc}^4(R^n)$. Then functions $V^{\frac{1}{2}}(x)u(x)$, $\frac{\partial^2 u(x)}{\partial x_i^2}$, $i = \overline{1, n}$ belong to the space $L_{2,\rho}(R^n)$.

Proof: Let φ be a test function such that a fixed non-negative function is a function that converges to one at. For any positive number let.

There is equality:

$$\langle f(x), \rho\varphi_\varepsilon(x)u(x)\rangle = \sum_{i=1}^n \langle \frac{\partial^4 u(x)}{\partial x_i^4}, \rho\varphi_\varepsilon u(x)\rangle + \langle V(x)u(x), \rho\varphi_\varepsilon u(x)\rangle, \quad \textbf{(11)}$$

where ⟨, ⟩ - are the scalar product in space $L_{2,\rho}(R^n)$.

Then, applying *Lemma 1* several times, we obtain from *Equality (11)*:

$$\begin{aligned}\langle f(x), \rho\varphi_\varepsilon(x)u(x)\rangle &= \sum_{i=1}^n \langle \frac{\partial^2 u(x)}{\partial x_i^2}, \frac{\partial^2 \rho}{\partial x_i^2}\varphi_\varepsilon u(x)\rangle + \sum_{i=1}^n \langle \frac{\partial^2 u(x)}{\partial x_i^2}, \rho\frac{\partial^2 \varphi_\varepsilon}{\partial x_i^2}u(x)\rangle \\ &+ \sum_{i=1}^n \langle \frac{\partial^2 u(x)}{\partial x_i^2}, \rho\varphi_\varepsilon\frac{\partial^2 u(x)}{\partial x_i^2}\rangle + 2\sum_{i=1}^n \langle \frac{\partial^2 u(x)}{\partial x_i^2}, \frac{\partial \rho}{\partial x_i}\frac{\partial \varphi_\varepsilon}{\partial x_i}u(x)\rangle \\ &+ 2\sum_{i=1}^n \langle \frac{\partial^2 u(x)}{\partial x_i^2}, \frac{\partial \rho}{\partial x_i}\varphi_\varepsilon\frac{\partial u(x)}{\partial x_i}\rangle + 2\sum_{i=1}^n \langle \frac{\partial^2 u(x)}{\partial x_i^2}, \rho\frac{\partial \varphi_\varepsilon}{\partial x_i}\frac{\partial u(x)}{\partial x_i}\rangle \\ &+ \langle V\big(x, u(x)\big)u(x), \varphi_\varepsilon(x)\rho u(x)\rangle.\end{aligned} \quad \textbf{(12)}$$

Equating the real parts of this equality, we have

$$\begin{aligned}\mathrm{Re}\langle f(x), \rho\varphi_\varepsilon(x)u(x)\rangle &= \sum_{i=1}^n \langle \frac{\partial^2 u(x)}{\partial x_i^2}, \rho\varphi_\varepsilon\frac{\partial^2 u(x)}{\partial x_i^2}\rangle + \langle V\big(x, u(x)\big)u(x), \varphi_\varepsilon(x)\rho u(x)\rangle \\ &+ A_{1,\varepsilon}(u) + A_{2,\varepsilon}(u) + 2A_{3,\varepsilon}(u) + 2A_{4,\varepsilon}(u) + 2A_{5,\varepsilon}(u),\end{aligned} \quad \textbf{(13)}$$

where

$$A_{1,\varepsilon}(u) = \mathrm{Re}\sum_{i=1}^n \langle \frac{\partial^2 u(x)}{\partial x_i^2}, \frac{\partial^2 \rho}{\partial x_i^2}\varphi_\varepsilon u(x)\rangle,$$

$$A_{2,\varepsilon}(u) = \mathrm{Re}\sum_{i=1}^n \langle \frac{\partial^2 u(x)}{\partial x_i^2}, \rho\frac{\partial^2 \varphi_\varepsilon}{\partial x_i^2}u(x)\rangle,$$

$$A_{3,\varepsilon}(u) = \mathrm{Re}\sum_{i=1}^n \langle \frac{\partial^2 u(x)}{\partial x_i^2}, \frac{\partial \rho}{\partial x_i}\frac{\partial \varphi_\varepsilon}{\partial x_i}u(x)\rangle,$$

$$A_{4,\varepsilon}(u) = \operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^2 u(x)}{\partial x_i^2}, \frac{\partial\rho}{\partial x_i}\varphi_\varepsilon\frac{\partial u(x)}{\partial x_i}\rangle,$$

$$A_{5,\varepsilon}(u) = \operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^2 u(x)}{\partial x_i^2}, \rho\frac{\partial\varphi_\varepsilon}{\partial x_i}\frac{\partial u(x)}{\partial x_i}\rangle.$$

We now evaluate the absolute values of the functionals $A_j(u), j = \overline{1,5}$,one by one.

Since the function $\varphi_\varepsilon(x)$ is real-valued, and

$$\left|\frac{\partial\varphi_\varepsilon}{\partial x_i}\right| \le M_0\varepsilon, \left|\frac{\partial^2\varphi_\varepsilon}{\partial x_i^2}\right| \le M_1\varepsilon^2, \text{for all } x \in R^n,$$

where

$$M_0 = \sup\left|\frac{\partial\varphi_\varepsilon}{\partial x_i}\right|, \qquad M_1 = \sup\left|\frac{\partial^2\varphi_\varepsilon}{\partial x_i^2}\right|.$$

Then $A_{2,\varepsilon}(u), A_{3,\varepsilon}(u)$ and $A_{5,\varepsilon}(u)$ tend to zero at $\varepsilon \to 0$. To evaluate $A_{1,\varepsilon}(u), A_{4,\varepsilon}(u)$, applying the inequality:

$$y_1 y_2 \le \frac{\alpha}{2}|y_1|^2 + \frac{1}{2\alpha}|y_2|^2, \qquad \textbf{(14)}$$

is valid for any $\alpha > 0$ and all y_1 и y_2, in view of *Inequalitie (6)* and *Inequalitie (7)*, we find the following estimates:

$$|A_{1,\varepsilon}(u)| \ge -\frac{\alpha\delta_1}{2}\sum_{i=1}^{n}\left\|\varphi_\varepsilon^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}, L_{2,\rho}(R^n)\right\|^2 - \frac{\delta_1}{2\alpha}\left\|\varphi_\varepsilon^{\frac{1}{2}}V^{\frac{1}{2}}u; L_{2,\rho}(R^n)\right\|^2, \qquad \textbf{(15)}$$

$$|A_{4,\varepsilon}(u)| \ge -\delta_2\sum_{i=1}^{n}\left\|\varphi_\varepsilon^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}, L_{2,\rho}(R^n)\right\|^2. \qquad \textbf{(16)}$$

Passing to the limit in *Equality (13)* at, using the Cauchy-Bunyakovsky inequality, at an arbitrary positive number, and constants from *Condition (8)* and *Condition (9)*, we find

$$\| f(x); L_{2,\rho}(R^n) \| \| u(x); L_{2,\rho}(R^n) \| \ge \operatorname{Re}\langle f(x), u(x)\rangle \ge$$

$$\left(1 - \frac{\delta_1\alpha}{2} - 2\delta_2\right)\sum_{i=1}^{n}\left\|\frac{\partial^2 u}{\partial x_i^2}; L_{2,\rho}(R^n)\right\|^2 + \left(1 - \frac{\delta_1}{2\alpha}\right)\left\|V^{\frac{1}{2}}u; L_{2,\rho}(R^n)\right\|^2.$$

This completes the proof of *Lemma 2*.

4 | Proof of Theorem

For any $\nu > 0$, the following equality holds

$$\langle f(x), \varphi_\varepsilon\rho f(x)\rangle = \langle Vu(x), \varphi_\varepsilon\rho Vu(x)\rangle + \nu\langle\sum_{i=1}^{n}\frac{\partial^4 u}{\partial x_i^4}, \varphi_\varepsilon\rho\frac{\partial^4 u}{\partial x_i^4}\rangle + A_1^\varepsilon(u) - A_2^\varepsilon(u), \qquad \textbf{(17)}$$

where

$$A_1^\varepsilon(u) = -(1+\nu)\operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^4 u}{\partial x_i^4}, \varphi_\varepsilon\rho Vu\rangle.$$

$$A_2^{\varepsilon}(u) = (1-\nu)\operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^4 u}{\partial x_i^4}, \varphi_\varepsilon \rho f(x)\rangle.$$

For any $\alpha_1 > 0$, the following inequality holds.

$$|A_2^{\varepsilon}(u)| \le \frac{\alpha_1|1-\nu|}{2}\sum_{i=1}^{n}\| \varphi_\varepsilon^{\frac{1}{2}}(x)\frac{\partial^4 u}{\partial x_i^4}\|^2 + \frac{|1-\nu|}{2\alpha_1}\| \varphi_\varepsilon^{\frac{1}{2}}(x)f(x)\|^2,$$

where $\|,\|$ - are norms in the space $L_{2,\rho}(R^n)$.

Therefore, we obtain

$$A_1^{\varepsilon}(u) = -(1+\nu)\operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^3 u}{\partial x_i^3}, \frac{\partial\varphi_\varepsilon}{\partial x_i}\rho V u(x)\rangle - (1+\nu)\operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^3 u}{\partial x_i^3}, \varphi_\varepsilon(x)\frac{\partial\rho}{\partial x_i}Vu(x)\rangle$$

$$-(1+\nu)\operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^3 u}{\partial x_i^3}, \varphi_\varepsilon(x)\rho\frac{\partial V}{\partial x_i}u(x)\rangle - (1+\nu)\operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^3 u}{\partial x_i^3}, \varphi_\varepsilon(x)\rho V\frac{\partial u(x)}{\partial x_i}\rangle.$$

Using *Lemma 1*, after simple transformations, we find

$$A_1^{\varepsilon}(u) = (1+\nu)\sum_{i=1}^{n}\langle\frac{\partial^2 u}{\partial x_i^2}, \varphi_\varepsilon(x)\rho V(x)\frac{\partial^2 u}{\partial x_i^2}(x)\rangle + B_1^{\varepsilon}(u) + B_2^{\varepsilon}(u) + B_3^{\varepsilon}(u) + 2B_4^{\varepsilon}(u) + 2B_5^{\varepsilon}(u) + 2B_6^{\varepsilon}(u) + 2B_7(\varepsilon) + 2B_8^{\varepsilon}(u) + 2B_9^{\varepsilon}(u), \quad \textbf{(18)}$$

where

$$B_1^{\varepsilon}(u) = (1+\nu)\operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^2 u}{\partial x_i^2}, \frac{\partial^2\varphi_\varepsilon}{\partial x_i^2}\rho V(x)u(x)\rangle,$$

$$B_2^{\varepsilon}(u) = (1+\nu)\operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^2 u}{\partial x_i^2}, \varphi_\varepsilon(x)\frac{\partial^2\rho}{\partial x_i^2}V(x)u(x)\rangle,$$

$$B_3^{\varepsilon}(u) = (1+\nu)\operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^2 u}{\partial x_i^2}, \varphi_\varepsilon(x)\rho\frac{\partial^2 V(x)}{\partial x_i^2}u(x)\rangle,$$

$$B_4^{\varepsilon}(u) = (1+\nu)\operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^2 u}{\partial x_i^2}, \frac{\partial\varphi_\varepsilon}{\partial x_i}\frac{\partial\rho}{\partial x_i}V(x)u(x)\rangle,$$

$$B_5^{\varepsilon}(u) = (1+\nu)\operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^2 u}{\partial x_i^2}, \frac{\partial\varphi_\varepsilon}{\partial x_i}\rho\frac{\partial V}{\partial x_i}(x)u(x)\rangle,$$

$$B_6^{\varepsilon}(u) = (1+\nu)\operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^2 u}{\partial x_i^2}, \frac{\partial\varphi_\varepsilon}{\partial x_i}\rho V(x)\frac{\partial u(x)}{\partial x_i}\rangle,$$

$$B_7^{\varepsilon}(u) = (1+\nu)\operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^2 u}{\partial x_i^2}, \varphi_\varepsilon(x)\rho\frac{\partial V(x)}{\partial x_i}\frac{\partial u(x)}{\partial x_i}\rangle,$$

$$B_8^{\varepsilon}(u) = (1+\nu)\operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^2 u}{\partial x_i^2}, \varphi_\varepsilon(x)\frac{\partial\rho}{\partial x_i}V(x)\frac{\partial u(x)}{\partial x_i}\rangle,$$

$$B_9^{\varepsilon}(u) = (1+\nu)\operatorname{Re}\sum_{i=1}^{n}\langle\frac{\partial^2 u}{\partial x_i^2}, \varphi_\varepsilon(x)\frac{\partial\rho}{\partial x_i}\frac{\partial V(x)}{\partial x_i}u(x)\rangle.$$

We evaluate alternately $B_j^{\varepsilon}(u)$, $j=\overline{1,9}$. Functionals $B_1^{\varepsilon}(u)$, $B_4^{\varepsilon}(u)$, $B_5^{\varepsilon}(u)$ and $B_6^{\varepsilon}(u)$ tend to zero at $\varepsilon \to 0$ by virtue of *Lemma 1*. When evaluating the functionals $B_2^{\varepsilon}(u)$, $B_3^{\varepsilon}(u)$, $B_7^{\varepsilon}(u)$, $B_8^{\varepsilon}(u)$ and $B_9^{\varepsilon}(u)$ for any $u(x) \in L_{2,\rho}(R^n)$, applying *inequalities (14)*, we obtain the following estimates:

$$|B_2^{\varepsilon}(u)| \le \frac{\beta\delta_1}{2}\sum_{i=1}^{n}\langle V^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}, \varphi_\varepsilon \rho V^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}\rangle + \frac{\delta_1}{2\beta}(Vu, \varphi_\varepsilon \rho Vu), \tag{19}$$

$$|B_3^{\varepsilon}(u)| \le \frac{\beta\sigma_1}{2}\sum_{i=1}^{n}\langle V^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}, \varphi_\varepsilon \rho V^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}\rangle + \frac{\sigma_1}{2\beta}(Vu, \varphi_\varepsilon \rho Vu), \tag{20}$$

$$|B_7^{\varepsilon}(u)| \le \sigma_2\sum_{i=1}^{n}\langle V^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}, \varphi_\varepsilon \rho V^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}\rangle. \tag{21}$$

$$|B_8^{\varepsilon}(u)| \le \sigma_3\sum_{i=1}^{n}\langle V^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}, \varphi_\varepsilon \rho V^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}\rangle, \tag{22}$$

$$|B_9^{\varepsilon}(u)| \le \frac{\beta\sigma_4}{2}\sum_{i=1}^{n}\langle V^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}, \varphi_\varepsilon \rho V^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}\rangle + \frac{\sigma_4}{2\beta}(Vu, \varphi_\varepsilon \rho Vu). \tag{23}$$

Here β is an arbitrary positive number, $\delta_1, \sigma_1, \sigma_2, \sigma_3$ and σ_4 - are constants from *Conditions (2)–(6)* respectively. Based on the obtained estimates from *Equality (17)*, taking into account inequality:

$$-|Z| \le \mathrm{Re}Z \le |Z|.$$

Go to the inequality:

$$\left\|\varphi_\varepsilon^{\frac{1}{2}} f(x))\right\|^2 \ge \| \varphi_\varepsilon^{\frac{1}{2}} Vu(x) \|^2 + \nu\sum_{i=1}^{n} \| \varphi_\varepsilon^{\frac{1}{2}}\frac{\partial^4 u}{\partial x_i^4}\|^2$$
$$-\frac{\alpha_1|1-\nu|}{2}\sum_{i=1}^{n}\| \varphi_\varepsilon^{\frac{1}{2}}\frac{\partial^4 u}{\partial x_i^4}\|^2 - \frac{|1-\nu|}{2\alpha_1}\| \varphi_\varepsilon^{\frac{1}{2}} f(x) \|^2$$
$$+(1+\nu)\left\{\sum_{i=1}^{n}\| \varphi_\varepsilon^{\frac{1}{2}} V^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}\|^2 - |B_2(\varepsilon)| - |B_3(\varepsilon)| - 2|B_7(\varepsilon) - 2|B_8(\varepsilon)| - 2|B_9(\varepsilon)|\right\},$$

where $\nu, \beta > 0, \alpha_1$ - are any positive numbers.

Further, having in mind *Enequalities (20)–(23)* for any $\nu, \alpha_1, \alpha > 0$, we obtain the inequality

$$\left\|\varphi_\varepsilon^{\frac{1}{2}} f(x))\right\|^2 \ge \| \varphi_\varepsilon^{\frac{1}{2}} Vu(x) \|^2 + \nu\sum_{i=1}^{n} \| \varphi_\varepsilon^{\frac{1}{2}}\frac{\partial^4 u}{\partial x_i^4}\|^2 -$$
$$-\frac{\alpha_1|1-\nu|}{2}\sum_{i=1}^{n}\| \varphi_\varepsilon^{\frac{1}{2}}\frac{\partial^4 u}{\partial x_i^4}\|^2 - \frac{|1-\nu|}{2\alpha_1}\| \varphi_\varepsilon^{\frac{1}{2}} f(x) \|^2$$
$$+(1+\nu)\left\{\sum_{i=1}^{n}\| \varphi_\varepsilon^{\frac{1}{2}} V^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}\|^2 -\right.$$
$$\left.-\left(\frac{\beta(\delta_1+\sigma_1+2\sigma_4)+4\sigma_2+4\sigma_3}{2}\right)\sum_{i=1}^{n}\| \varphi_\varepsilon^{\frac{1}{2}} V^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2}\|^2 - \frac{\delta_1+\sigma_1+2\sigma_4}{2\beta}\quad \| \varphi_\varepsilon^{\frac{1}{2}} Vu(x) \|^2\right\}.$$

Passing to the limit at $\varepsilon \to 0$, we obtain

$$(1+\frac{|1-\nu|}{2\alpha_1})||f(x))||^2 \geq (1-\frac{(1+\nu)(\delta_1+\sigma_1+2\sigma_4)}{2\beta}) \parallel Vu(x) \parallel^2$$
$$+(\nu-\frac{\alpha_1|1-\nu|}{2})\sum_{i=1}^{n} \parallel \frac{\partial^4 u}{\partial x_i^4} \parallel^2$$
$$+(1+\nu)\left(1-\frac{\beta(\delta_1+\sigma_1+2\sigma_4)+4\sigma_2+4\sigma_3}{2}\right)\sum_{i=1}^{n} \parallel V^{\frac{1}{2}}\frac{\partial^2 u}{\partial x_i^2} \parallel^2.$$

Hence, the following is obvious: if $|1-\nu| < 2\alpha_1$, $\alpha_1|1-\nu| < 2\nu$, $(1+\nu)(\delta_1+\sigma_1+2\sigma_4) < 2\beta$ and $\beta(\delta_1 + \sigma_1+2\sigma_4)+4\sigma_2+4\sigma_3 < 2$, then the *Coercive Estimate (6).* The existence of the numbers $\alpha_1, \beta, \nu > 0$ is deduced from *Inequality (8).*

The separability of the *Nonlinear Operator (1)* in the space $L_{2,\rho}(R^n)$ follows from the coercive *Inequality (12).*

This completes the proof of the theorem.

5 | Conclusion

In this work, we investigated the coercive properties and separability of a fourth-order differential operator in a weighted function space. Through the establishment of coercive inequalities, we derived sufficient conditions that ensure the separability of the operator under consideration. These results build upon and extend existing research on second-order and biharmonic operators, providing a broader theoretical foundation for the analysis of higher-order differential equations. The developed criteria contribute significantly to the theory of elliptic differential operators, particularly in weighted Hilbert spaces, where traditional techniques may not be directly applicable. The findings also highlight the importance of weight functions and function classes in determining operator behavior. This study opens the door for future research on nonlinear operators, matrix potentials, and boundary value problems in complex domains, thereby deepening our understanding of the interplay between operator theory and functional analysis.

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Author Contribution

Conceptualization, O.K. and J.N.; Methodology, O.K.; Software, O.K.; Validation, O.K. and J.N.; formal analysis, O.K.; investigation, O.K.; resources, O.K.; data maintenance, J.N.; writing-creating the initial design, J.N.; writing-reviewing and editing. All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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