Karshi Multidisciplinary International Scientific Journal

www.kmisj.reapress.com

K. Multidiscip. Int. Sci. J. Vol. 2, No. 1 (2025) 28–39.

Paper Type: Original Article

Zeros of the Riemann Zeta Function on Short Intervals of the Critical Line

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Citation:

Received: 14 November 2024 *Revised: 28 January 2025* *Accepted: 18 May 2025*	Khayrulloev, Sh. (2025). Zeros of the Riemann zeta function on short intervals of the critical line. *Karshi multidisciplinary international scientific journal*, *2*(1), 28-39.

Abstract

This paper presents a study of the distribution of zeros of the Riemann zeta function, specifically focusing on those of odd order, within short intervals along the critical line. The Riemann zeta function plays a central role in analytic number theory, and its nontrivial zeros are deeply connected with the distribution of prime numbers. The critical line, where the real part of the argument is one-half, is of particular interest due to the famous Riemann Hypothesis, which suggests that all nontrivial zeros lie on this line. Although the hypothesis remains unproven, considerable progress has been made in understanding the behavior and density of these zeros. The objective of this study is to examine the frequency and location of odd-order zeros in small neighborhoods on the critical line. We build upon earlier foundational work by mathematicians such as Hardy, Littlewood, Selberg, and Karatsuba. Their contributions established the groundwork for understanding the occurrence of zeros in specific ranges, and this paper aims to refine and extend those results. In particular, we aim to verify a hypothesis that proposes the presence of a significant number of such zeros in very short intervals. To achieve this, we utilize analytic methods involving trigonometric sums and exponential pair techniques. These approaches allow us to estimate the relevant quantities without requiring explicit evaluation of the zeros themselves. The method employed in this paper refines previous bounds and enables the detection of zeros in intervals that are shorter than those considered in earlier works. Additionally, this study provides sharper criteria under which the existence of such zeros can be guaranteed. Our findings support the hypothesis that zeros of odd order are not only present but relatively frequent in short segments along the critical line. This contributes valuable insight into the fine-scale structure of the zeta function's zeros and affirms the robustness of analytic techniques based on exponential sums. Moreover, the results have broader implications in number theory, particularly in areas concerned with prime number theorems and related analytic functions.

Keywords: Riemann zeta function, Exponential pair, Critical line, Trigonometric sum, Odd-order zeros, Nontrivial estimate, Absolute constant, Selberg sum.

1 | Introduction

The Riemann zeta function stands at the heart of analytic number theory, representing one of the most profound and extensively studied objects in mathematics. Originally introduced by Bernhard Riemann in the

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https://doi.org/10.22105/kmisj.v2i1.83

mid-nineteenth century, the function encodes deep information about the distribution of prime numbers. Among the most intriguing aspects of this function are its nontrivial zeros, whose behavior governs critical results in prime number theory and beyond. The famous Riemann Hypothesis proposes that all nontrivial zeros lie along a specific vertical line in the complex plane, known as the critical line. Despite extensive numerical verification and theoretical exploration, this hypothesis remains unproven. Interest in the zeros of the zeta function, particularly those on the critical line, has driven a century of mathematical research. Early advances by Hardy demonstrated that an infinite number of such zeros exist. Subsequent contributions by mathematicians such as Littlewood, Selberg, and Karatsuba have gradually revealed more detailed information about the location, density, and nature of these zeros.

One aspect that has received increasing attention is the distribution of zeros within short intervals along the critical line. The challenge lies in estimating how many zeros, especially of odd order, can be found in very small segments, and under what conditions their existence can be guaranteed. This paper aims to contribute to this ongoing investigation by refining previously established results concerning the presence of odd-order zeros in short intervals. The significance of focusing on odd-order zeros lies in their close connection to the structure of the Hardy Z-function, which provides a real-valued representation of the zeta function along the critical line. Understanding the distribution of these specific zeros can offer more precise insights into the behavior of the zeta function and potentially lead to advancements in related conjectures. To address this, we apply advanced techniques based on the theory of exponential sums. These tools allow us to estimate the frequency and distribution of zeros indirectly, bypassing the need for explicit computation of their locations. The exponential pair method, in particular, has proven highly effective in previous studies and forms a core component of our approach. By refining these methods, we aim to establish improved bounds on the number of odd-order zeros within short segments and verify certain long-standing hypotheses under more general conditions. This work not only deepens our understanding of the Riemann zeta function but also contributes to broader developments in number theory. The techniques and results discussed herein may have implications for other functions with similar analytic structures and could influence future efforts to resolve the Riemann Hypothesis itself. Through this study, we continue the long tradition of exploring the fascinating and intricate world of zeta-function zeros.

The zeros of the Riemann zeta-function on short intervals of the critical line are both an interesting and challenging study in analytic number theory. The Riemann zeta-function $\zeta(s)$, regularly analytically extended to the entire complex plane except for a single special point $s = 1$, has been the basis for many essential conclusions in analytic number theory.

The study of zeros of the Riemann zeta function on short intervals of the critical line opens new possibilities for a deeper study of prime numbers and their properties. This is an actively developing and topical direction in mathematics, which continues to attract the attention of a wide range of mathematical scientists who can expand our knowledge and continue the path for future discoveries.

The eminent German mathematician Riemann [1] formulated a hypothesis assuming that all nontrivial zeros of the Riemann zeta function are on a "critical line". Despite many studies and experiments, this hypothesis still remains unproven.

The first important result related to the location of zeros of the zeta function on the critical line was the theorem proved by Hardy [2]. He managed to prove that the number of such zeros is infinite. This discovery was a significant step in understanding the properties of the Riemann zeta function and paved the way for further research in this area.

Hardy et al. [3] proved the following statement: for any positive value of, there exists such T_0, dependent on ε, greater than zero, for all T, greater than T_0, $H \geqslant T^{\frac{1}{4}+\varepsilon}$ the interval $(T, T + H)$ contains an odd-order zero of the function $\zeta\left(\frac{1}{2} + it\right)$. It follows that the interval $(0, T)$ contains more than $T^{\frac{3}{4}+\varepsilon}$ zeros of odd order of the function $\zeta\left(\frac{1}{2} + it\right)$.

The number of zeros of the function, lying on the interval, we denote by. We recall the significant contributions of G. Hardy and D. Littlewood carried out in 1921 [4]. In the course of their research, these scientists proved the following theorem: For any there exists such that at, the inequality is true.

$$N_0(T+H)-N_0(T)\geqslant cH.$$

In 1942, the eminent mathematician Atle Selberg [5] successfully proved an amplified version of Hardy and Littlewood's theorem, which has a special significance, i.e., if the conditions of Hardy and Littlewood's theorem are satisfied, the inequality is true:

$$N_0(T+H)-N_0(T)\geqslant cH\ln T. \quad (1)$$

In A.Selberg's evaluation of *Inequality (1)*, an interesting hypothesis arose that *Inequality (1)* can be fulfilled also at smaller values of H, i.e., at $H=T^{\alpha+\varepsilon}$, where α is a fixed positive number smaller than $1/2$ [5].

In 1976, Czech mathematician Changa et al. [6] obtained a new result in the above problem: If $T\geqslant T_0>0$, $H\geqslant T^{5/12}\ln^3 T$ is true the inequality

$$N_0(T+H)-N_0(T)\geqslant cH,$$

$c>0$ - absolute constant.

In 1984, the outstanding mathematician Eremin et al. [7] proved Selberg's hypothesis at $\alpha=27/82$, that is, he proved the following theorem: Let ε be an arbitrary positive number not exceeding 0.001, $T\geqslant T_0>0$, $H\geqslant T^{27/82+\varepsilon}$. Then there exists a positive constant $c=c(\varepsilon)$ such that

$$N_0(T+H)-N_0(T)\geqslant cH\ln T.$$

A.A.Karatsuba made a fascinating statement about the number $\alpha=27/82$, which can be replaced by a smaller number [7]. However, we should pay attention to the fact that it is connected with very complicated evaluations of a special kind of trigonometric sums.

2 | Formulation of the Main Result

In the present paper, applying the exponential pair method [8], following the works [9–11], we prove A. Selberg's hypothesis when $\alpha=1515/4816$.

The following theorem is valid.

Theorem 1. Let (κ,λ) be an arbitrary exponential pair,

$$\theta(\kappa,\lambda)=\frac{\kappa+\lambda}{2\kappa+2},$$

ε- is an arbitrary positive number not exceeding 0.001, $T\geqslant T_0(\varepsilon)>0$, $H=T^{\theta(\kappa,\lambda)+\varepsilon}$. Then there exists a positive constant $c=c(\varepsilon)$ such

$$N_0(T+H)-N_0(T)\geqslant cH\ln T.$$

Note that the exponent $\theta(\kappa,\lambda)$ in *Theorem 1* was previously considered in the Gauss problem on the number of integer points in a circle $x^2+y^2\leqslant R$, as well as in the evaluation of the residual term in the Dirichlet divisor problem on the number of integer points in a hyperbola $xy\leqslant N$, $x>0$, $y>0$. The best estimate from above for $\theta(\kappa,\lambda)$ so far has been obtained by Bourgain and Watt [12]. They proved that

$$\theta_0=\min_{\kappa,\lambda\in\mathcal{P}}\theta(\kappa,\lambda)=\min_{\kappa,\lambda\in\mathcal{P}}\frac{\kappa+\lambda}{2\kappa+2}\leqslant\frac{1515}{4816}=\frac{1}{3}-\frac{271}{3\cdot 4816}\approx 0.314576,$$

where $\mathcal{P}$ is the set of all exponential pairs.

The following follows from [12] and from *Theorem 1*.

Corollary. Let ε be an arbitrary positive number not exceeding 0.001, $T \geqslant T_0(\varepsilon) > 0$, $H = T^{\frac{1515}{4816}+\varepsilon}$. Then there exists a positive constant $c = c(\varepsilon)$ such that

$$N_0(T+H) - N_0(T) \geqslant cH\ln T.$$

3 | Proof of Theorem 1

Let $X = T^{0.01\varepsilon}$. Consider the Hardy-Selberg function $F(t)$, at $T \leqslant t \leqslant T+H$:

$$F(t) = e^{i\theta(t)}\zeta(0{,}5+it)\left|\varphi\left(\frac{1}{2}+it\right)\right|^2, \quad e^{i\theta(t)} = \frac{\pi^{-it/2}\Gamma(\frac{1}{4}+\frac{it}{2})}{\left|\Gamma(\frac{1}{4}+\frac{it}{2})\right|},$$

$$\varphi\left(\frac{1}{2}+it\right) = \sum_{\nu\leqslant X}\frac{\beta(\nu)}{\sqrt{\nu}}\nu^{-it}, \quad \beta(\nu) = \begin{cases}\alpha(\nu)\left(1-\frac{\ln\nu}{\ln X}\right), & 1 \leqslant \nu < X,\\ 0, & \nu \geqslant X,\end{cases}$$

and real numbers $\alpha(\nu)$ are determined from the equality

$$\frac{1}{\sqrt{\zeta(s)}} = \sum_{\nu=1}^{\infty}\frac{\alpha(\nu)}{\nu^s}, \quad \text{Re}s > 1.$$

It follows from the definition of the function $F(t)$ and the functional equation $\zeta(s)$ that the function $F(t)$ takes on valid values at valid t, and the valid zeros $F(t)$ of odd order are the valid zeros of the odd order function $\zeta(0{,}5+it)$.

Assume that, where is a constant, the value will be specified in Section 3. We denote by the symbol a subset of the interval, on which the inequality is satisfied

$$\int_t^{t+h}|F(u)|du > \left|\int_t^{t+h}F(u)du\right|, \quad t \in E.$$

Since outside E, these integrals are equal, then

$$\int_T^{T+H}\int_t^{t+h}|F(u)|dudt \leqslant \int_E dt\left(\int_t^{t+h}|F(u)|du\right) + \int_T^{T+H}\left|\int_t^{t+h}F(u)du\right|dt.$$

Using Cauchy's inequality, we can obtain the following relation:

$$I_1 \leqslant \sqrt{\mu(E)I_2} + \sqrt{HI_3},$$

where

$$I_1 = \int_T^{T+H}\int_t^{t+h}|F(u)|dudt, \quad I_2 = \int_T^{T+H}\left(\int_t^{t+h}|F(u)|du\right)^2dt, \quad I_3 = \int_T^{T+H}\left|\int_t^{t+h}F(u)du\right|^2dt.$$

Evaluating from below the integral I_1 and from above the integrals I_2, I_3, we obtain the following evaluation from below for the measure of the set E:

$$\mu(E) > c_1H, \quad c_1 > 0,$$

hence, the validity of the theorem is deduced.

Evaluating the integral I_1 from below

First of all, we derive the following relations sequentially:

$$I_1 = \int_T^{T+H}\int_t^{t+h} |F(u)|dudt \geqslant h\int_{T+h}^{T+H} |F(u)|du \geqslant h\left|\int_{T+h}^{T+H} \zeta\left(\frac{1}{2}+it\right)\varphi^2\left(\frac{1}{2}+it\right)dt\right|. \quad (3)$$

By Γ we denote a rectangle with vertices: $0{,}5 + i(T + h)$, $2 + i(T + h)$, $2 + i(T + H)$, $0{,}5 + i(T + H)$. In this rectangle, the function $\zeta\left(\frac{1}{2}+it\right)\varphi^2\left(\frac{1}{2}+it\right)$ is analytic and single-valued. Then, by Cauchy's theorem, the following holds

$$\int_\Gamma \zeta(s)\varphi^2(s)ds = 0,$$

which can be represented in the following form:

$$\int_{T+h}^{T+H} \zeta\left(\frac{1}{2}+it\right)\varphi^2\left(\frac{1}{2}+it\right)dt = \int_{T+h}^{T+H} \zeta(2+it)\varphi^2(2+it)dt -$$
$$-i\int_{\frac{1}{2}}^{2} \zeta\big(\sigma + i(T+H)\big)\varphi^2\big(\sigma + i(T+H)\big)d\sigma + i\int_{\frac{1}{2}}^{2} \zeta\big(\sigma + i(T+h)\big)\varphi^2\big(\sigma + i(T+h)\big)d\sigma. \quad (4)$$

Using the definitions of the function $\varphi(s)$, at $\mathrm{Re}s > 1$, we have

$$\zeta(s)\varphi^2(s) = \sum_{n=1}^{\infty}\sum_{\nu_1<X}\sum_{\nu_2<X}\frac{\beta(\nu_1)\beta(\nu_2)}{(n\nu_1\nu_2)^s} = \sum_{n=1}^{\infty}\frac{a(n)}{n^s} = 1 + \sum_{n=2}^{\infty}\frac{a(n)}{n^s},$$

and $|a(n)| \leqslant \tau_3(n)$. Hence, it follows

$$\int_{T+h}^{T+H} \zeta(2+it)\varphi^2(2+it)dt = H - h + O\left(\sum_{n=2}^{\infty}\frac{a(n)}{n^2\ln n}\right) = H - h + O(1). \quad (5)$$

At $\sigma \geqslant 1/2$, using estimates

$$|\varphi(s)| \leqslant 2\sqrt{X}, \quad \zeta(\sigma+it) = O\big(t^{1/6}\ln t\big), \quad \varphi(\sigma+it) = O(\sqrt{X}),$$

we get

$$\int_{\frac{1}{2}}^{2} \zeta\big(\sigma + i(T+H)\big)\varphi^2\big(\sigma + i(T+H)\big)d\sigma = O\left(T^{\frac{1}{6}}X\ln T\right), \quad (6)$$

$$\int_{\frac{1}{2}}^{2} \zeta\big(\sigma + i(T+h)\big)\varphi^2\big(\sigma + i(T+h)\big)d\sigma = O\left(T^{\frac{1}{6}}X\ln T\right). \quad (7)$$

Substituting the found *Estimates (5)-(7)* into *Estimate (4)*, then into *Estimate (3)*, we have the following estimate from below for the integral I_1:

$$I_1 \geqslant hH - h^2 + O\big(T^{1/6}X\ln T\big).$$

To evaluate the integrals from above I_2 and I_3 we use the functional equation of the Hardy-Selberg function $F(t)$, $T \leqslant t \leqslant T + H$, $X = T^{0.001\varepsilon}$

$$F(t) = F_1(t) + \overline{F_1(t)} + O\big(t^{-1/4}X^2\ln t\big),$$

where

$$F_1(t) = e^{i\theta_1(t)} \sum_{\lambda<\sqrt{t/(2\pi)}} \frac{a(\lambda)}{\sqrt{\lambda}} \lambda^{-it}, \quad \theta_1(t) = t\ln\sqrt{\frac{t}{2\pi}} - \frac{t}{2} - \frac{\pi}{8}, \quad a(\lambda) = \sum_{n\nu_2/\nu_1=\lambda} \frac{\beta(\nu_1)\beta(\nu_2)}{\nu_1},$$

$\nu_1, \nu_2 < X$, λ are rational numbers whose denominator does not exceed X. Replacing $P_1 = \sqrt{t/(2\pi)}$ by $P = \sqrt{T/(2\pi)}$, and $\theta_1(t)$ by $\theta(t) = t\ln P - T/2 - \pi/8$ we find

$$F_1(t) = F_0(t) + R_1 + R_2,$$

where

$$F_0(t) = e^{i\theta(t)} \sum_{\lambda<P} \frac{a(\lambda)}{\sqrt{\lambda}} \lambda^{-it}, \quad R_1 \ll \frac{H^2}{T} \sum_{\lambda<P_1} \frac{a(\lambda)}{\sqrt{\lambda}} \lambda^{-it}, \quad R_2 \ll \left| \sum_{P\leqslant\lambda\leqslant P_1} \frac{a(\lambda)}{\sqrt{\lambda}} \lambda^{-it} \right|.$$

Trivially evaluating the sums R_1 and R_2, we

$$F_1(t) = F_0(t) + O(\ln^{-5} T).$$

Thus, we have

$$F(t) = F_0(t) + \overline{F_0(t)} + O(\ln^{-5} T). \tag{8}$$

Assume that satisfies the equality, where is an integer, then

$$F_0(t) = e^{il\ln P} \sum_{\lambda<P} \frac{a(\lambda)}{\sqrt{\lambda}} \lambda^{-it}.$$

Evaluation of the integral I_2 from above

To the internal integral, applying Cauchy's inequality and then *Formula (8)*, we find

$$I_2 \leqslant h^2 \int_T^{T+H+h} \left|F_0(u) + \overline{F_0(u)} + O(\log^{-5} T)\right|^2 du \ll h^2(J + H\mathcal{L}^{-10}),$$

where

$$J = \int_T^{T+H_1} |F_0(t)|^2 dt, \quad H_1 = H + h, \quad \mathcal{L} = \ln T.$$

Now, let's evaluate from above the integral J. We have

$$J = \int_0^{H_1} |F_0(T+t)|^2 dt \leqslant e \sum_{\lambda_1<P} \sum_{\lambda_2<P} \frac{a(\lambda_1)a(\lambda_2)}{\sqrt{\lambda_1\lambda_2}} \left(\frac{\lambda_1}{\lambda_2}\right)^{iT} \int_{-\infty}^{\infty} \exp\left(-\left(\frac{t}{H_1}\right)^2 + it\ln\frac{\lambda_1}{\lambda_2}\right) dt.$$

Using the formula

$$\int_{-\infty}^{\infty} \exp(-t^2 - i\alpha t) dt = \sqrt{\pi} \exp\left(-\left(\frac{\alpha}{2}\right)^2\right),$$

at real value α, we have

$$J \leqslant e\sqrt{\pi} H_1 \sum_{\lambda_1<P} \sum_{\lambda_2<P} \frac{a(\lambda_1)a(\lambda_2)}{\sqrt{\lambda_1\lambda_2}} \left(\frac{\lambda_1}{\lambda_2}\right)^{iT} \exp\left(-\left(\frac{H_1}{2}\ln\frac{\lambda_1}{\lambda_2}\right)^2\right).$$

Representing the last twofold as the sum of two summands, one of which occurs when $\lambda_1 = \lambda_2$, we obtain the estimate of

$$J \ll H_1(\Sigma_0 + W_0),$$

where

$$\Sigma_0 = \sum_{\lambda<P} \frac{a^2(\lambda)}{\lambda}, \quad W_0 = \left| \sum_{\lambda_1<P} \sum_{\lambda_2<P} \frac{a(\lambda_1)a(\lambda_2)}{\sqrt{\lambda_1\lambda_2}} \left(\frac{\lambda_1}{\lambda_2}\right)^{iT} \exp\left(-\left(\frac{H_1}{2}\ln\frac{\lambda_2}{\lambda_1}\right)^2\right) \right|.$$

Taking into account [7], for Σ_0, we obtain the following estimate:

$$\Sigma_0 \ll \frac{\ln P}{\ln X}. \qquad \textbf{(9)}$$

To estimate the sum of W_0 and other similar sums that appear below, we will consider the following sum of the form

$$W = \left| \sum_{\lambda_1<\lambda_2<P} \frac{a(\lambda_1)a(\lambda_2)}{\sqrt{\lambda_1\lambda_2}} \left(\frac{\lambda_2}{\lambda_1}\right)^{iT} B(\lambda_1)\overline{B(\lambda_2)} e^{-\left(\frac{H_1}{2}\ln\frac{\lambda_2}{\lambda_1}\right)^2} \right|,$$

where $H_1 = H + h$ or $H_1 = H$ and $B(\lambda)$ is an arbitrary complex number with the condition $|B(\lambda)| \leqslant B$.

Let us fix $\nu_1, \nu_2, \nu_3, \nu_4$; let $\frac{\nu_2\nu_3}{\nu_1\nu_4} = \frac{a}{b}$, $(a, b) = 1$. Assuming $N = \Lambda\nu_2/\nu_1$, $N_1 = \Lambda_1\nu_2/\nu_1$, consider the sum of W_1 over n_1, n_2,

$$W_1 = \left| \sum_{N<n_1\leqslant N_1} \sum_{n_1ba^{-1}<n_2\leqslant n_1ba^{-1}(1+H^{-1}\mathcal{L})} \frac{1}{\sqrt{n_1n_2}} \left(\frac{n_2a}{n_1b}\right)^{iT} B\left(\frac{n_1\nu_1}{\nu_2}\right) \overline{B\left(\frac{n_2\nu_3}{\nu_4}\right)} e^{-\left(\frac{H_1}{2}\ln\frac{n_2a}{n_1b}\right)^2} \right|.$$

The following inequality is true:

$$W(\Lambda) \leqslant \sum_{\nu_1,\nu_2,\nu_3,\nu_4} \frac{1}{\sqrt{\nu_1\nu_2\nu_3\nu_4}} W_1.$$

Next, we estimate W_1. We have

$$W_1 \leqslant \sum_{0\leqslant r_1<a} \sum_{0\leqslant r_2<b} W_2,$$

where

$$W_2 \leqslant \frac{1}{\sqrt{ab}} \sum_{0\leqslant h\leqslant H_2} \left| \sum_{M<m\leqslant M_1} E(m, h) \left(\frac{m + h + \xi_3}{m + \xi_2}\right)^{iT} \right|, \qquad \textbf{(10)}$$

where the new symbols have been introduced:

$$E(m\ h) = \frac{B_1(m_1)B_2(m;h)}{\sqrt{(m+\xi_2)(m+h+\xi_3)}} e^{-\left(\frac{H_1}{2}\ln\frac{m+h+\xi_3}{m+\xi_2}\right)^2},$$

$$H_2 = 2N_1a^{-1}H^{-1}\mathcal{L} \quad \xi_2 = r_1a^{-1} \quad \xi_3 = r_2b^{-1} \quad Na^{-1} < M \leqslant M_1 \leqslant N_1a^{-1},$$

In *Inequality (10)*, we apply Abel transformations to the inner sum over m and find

$$\sum_{M<m\leqslant M_1} E(m,h)\left(\frac{m+h+\xi_3}{m+\xi_2}\right)^{iT} = -\int\limits_{M}^{M_1} C(u)E_u{}'(u,h)du + E(M_1,h)C(M_1),$$

where

$$C(u) = \sum_{M<m\leqslant u} e\left(\frac{T}{2\pi}\ln\frac{m+h+\xi_3}{m+\xi_2}\right), \quad |E(u,h)| \leqslant \frac{B_0^2}{u}.$$

Since the function $E(u,h)$ is piecewise monotone, passing to the estimate (10), we find

$$\left|\sum_{M<m\leqslant M_1} E(u,h)\left(\frac{m+h+\xi_3}{m+\xi_2}\right)^{iT}\right| \ll \frac{B_0^2}{M}\max_{u\leqslant M_1}|C(u)|.$$

Therefore, we get

$$W_2 \leqslant \frac{B_0^2}{M\sqrt{ab}}\sum_{0\leqslant h\leqslant H_2}\left|\sum_{M<m\leqslant u} e\left(\frac{T}{2\pi}\ln\frac{m+h+\xi_3}{m+\xi_2}\right)\right|.$$

Estimating the sum of $C(u)$. To estimate the sum of $C(u)$, we will use the exponential pair method [8]. Suppose,

$$f(u) = \frac{T}{2\pi}\ln\frac{u+h+\xi_3}{u+\xi_2}, \quad B = u - M \leqslant M_1 - M \leqslant M, \quad A = \frac{T|h+\xi_3-\xi_2|}{M^2}.$$

Find the derivative k -th order of the function $f(u)$ $(k = 1,2,\ldots)$:

$$f^{(k)}(u) = \frac{(-1)^k(k-1)!\,T(h+\xi_3-\xi_2)}{2\pi}\sum_{j=0}^{k-1}\frac{1}{(u+\xi_2)^{k+j}(u+h+\xi_3)^{k-j}}, \quad k = 1,2,\ldots$$

Therefore

$$AB^{1-k} \ll f^{(k)}(u) \ll AB^{1-k}.$$

Hence, for any exponential pair (κ,λ), we have

$$W_2 \ll \frac{B_0^2}{M\sqrt{ab}}\sum_{0\leqslant h\leqslant H_2}\left(\frac{T|h+\xi_1-\xi_2|}{M^2}\right)^{\kappa} M^{\lambda}.$$

Keeping in mind that $|h+\xi_1-\xi_2| \ll H_2 \ll Na^{-1}H^{-1}L$, $Na^{-1} < M \leqslant 2Na^{-1}$, we get

$$W_2 \ll \frac{B_0^2}{\sqrt{ab}}M^{-1-2\kappa+\lambda}T^{\kappa}H_2^{\kappa+1} \ll \frac{B_0^2}{\sqrt{ab}}N^{-\kappa+\lambda}a^{\lambda-\kappa}T^{\kappa}H^{-\kappa-1}\mathcal{L}^{\kappa+1}.$$

Substituting the found estimate into the expression for W_1, we have

$$W_1 \leqslant \sum_{0\leqslant r_1<a}\sum_{0\leqslant r_2<b}\frac{B_0^2}{\sqrt{ab}}N^{-\kappa+\lambda}a^{\lambda-\kappa}T^{\kappa}H^{-\kappa-1}\mathcal{L}^{\kappa+1} \ll B_0^2N^{-\kappa+\lambda}a^{0,5+\lambda-\kappa}b^{0,5}T^{\kappa}H^{-\kappa-1}\mathcal{L}^{\kappa+1}.$$

Given that $N = \frac{\Lambda\nu_2}{\nu_1}, \frac{a}{b} = \frac{\nu_2\nu_3}{\nu_1\nu_4}$, $0,5+\lambda-\kappa > 0.5$, we find

$$W_1 \ll \nu_1^{0,5+\kappa-\lambda}\nu_2^{2(\lambda-\kappa)+0,5}\nu_3^{0,5+\lambda-\kappa}\nu_4^{0,5}T^{\kappa}H^{-\kappa-1}\mathcal{L}^{\kappa+1}B_0^2\Lambda^{\lambda-\kappa}.$$

Substituting the found estimate into the expression for $W(\Lambda)$, and summing over $\nu_1, \nu_2, \nu_3, \nu_4$, we obtain

$$W(\Lambda) \ll B_0^2\Lambda^{\lambda-\kappa}T^{\kappa}H^{-\kappa-1}X^{4+2(\lambda-\kappa)}\mathcal{L}^{\kappa+1}.$$

Since $HX^{-2}\mathcal{L}^{-1} < \Lambda < P = \sqrt{\frac{T}{2\pi}}$ and $0 \leqslant \lambda - \kappa \leqslant 1$, then

$$W(\Lambda) \ll B_0^2 T^{\frac{\lambda-\kappa}{2}+\kappa} H^{-\kappa-1} X^{4+2(\lambda-\kappa)} \mathcal{L}^{\kappa+1} \ll B_0^2 T^{\frac{\kappa+\lambda}{2}} H^{-\kappa-1} X^7.$$

According to the condition of the theorem, $H = T^{\frac{\kappa+\lambda}{2(\kappa+1)}+\varepsilon}$ and by the definition of exponential pairs $0 \leqslant \kappa \leqslant 0{,}5$, therefore, we find

$$W(\Lambda) \ll B_0^2 \left(\frac{T^{\frac{\kappa+\lambda}{2(\kappa+1)}+\frac{0{,}07\varepsilon}{\kappa+1}}}{H} \right)^{\kappa+1} \ll B_0^2 T^{-0{,}93\varepsilon}.$$

Substituting the found estimate into the expression for W, and taking into account the definition of the values B and B_0 we have

$$W \ll \mathcal{L} W(\Lambda) + B^2 e^{-0.1\mathcal{L}^2} \ll B_0^2 T^{-0{,}93\varepsilon} \mathcal{L} + B^2 e^{-0.1\mathcal{L}^2} \ll B T^{-0.9\varepsilon}. \quad \textbf{(11)}$$

Thus, to estimate the double sum W_0, assuming in the definition W, $B(\lambda_1) = 1$, $B_0 = \text{const}$, and Considering *Eq. (11)*, we obtain:

$$W_0 \ll T^{-0.9\varepsilon}.$$

Thus, we find an estimate of the integral I_2:

$$I_2 \ll h^2 (H_1(\Sigma_0 + W_0) + H\mathcal{L}^{-10}) \ll h^2 \left(H_1 \frac{\ln P}{\ln X} + H_1 T^{-0{,}9\varepsilon} + H\mathcal{L}^{-10} \right) \leqslant c(\varepsilon) h^2 H_1.$$

Estimation of the integral I_3 from above

Similarly to the evaluation of I_2, applying *Relation (8)*, we arrive at the inequality

$$I_3 = \int_T^{T+H} \left| \int_t^{t+h} F(u)du \right|^2 dt \ll J + Hh^2\mathcal{L}^{-10},$$

$$J = \int_T^{T+H} \left| \int_t^{t+h} F_0(u)du \right|^2 dt, \quad F_0(u) = e^{iu\ln P} \sum_{\lambda<P} \frac{a^2(\lambda)}{\sqrt{\lambda}} \lambda^{-iu}.$$

Let the existing positive number ε_1 not exceed 0.1, its more precise value will be specified later. Dividing the summation in $F_0(u)$ by the parameter λ into two parts: $\lambda < P^{1-\varepsilon_1}$, $P^{1-\varepsilon_1} \leqslant \lambda < P$, we obtain the following relation

$$J \ll J_1 + J_2,$$

where

$$J_1 = \int_T^{T+H} \left| \int_t^{t+h} \sum_{\lambda<P^{1-\varepsilon_1}} \frac{a(\lambda)}{\sqrt{\lambda}} \left(\frac{P}{\lambda}\right)^{iu} du \right|^2 dt, \quad J_2 = \int_T^{T+H} \left| \int_t^{t+h} \sum_{P^{1-\varepsilon_1} \leqslant \lambda < P} \frac{a(\lambda)}{\sqrt{\lambda}} \left(\frac{P}{\lambda}\right)^{iu} du \right|^2 dt.$$

For the integral J_1, we obtain the following estimate:

$$J_1 \leqslant e \int_{-\infty}^{\infty} e^{-(t/H)^2} \left| \sum_{\lambda<P^{1-\varepsilon_1}} \frac{a(\lambda)}{\sqrt{\lambda}} \left(\frac{P}{\lambda}\right)^{i(T+H)} \frac{(P/\lambda)^{ih}-1}{\ln(P/\lambda)} \right|^2 dt \ll H(\Sigma_1 + W_1),$$

$$\Sigma_1 = \sum_{\lambda<P^{1-\varepsilon_1}} \frac{a^2(\lambda)}{\lambda \ln^2(P/\lambda)}, \quad W_1 = \left| \sum_{\lambda_1<\lambda_2<P^{1-\varepsilon_1}} \frac{a(\lambda_1)a(\lambda_2)}{\sqrt{\lambda_1\lambda_2}} \left(\frac{\lambda_2}{\lambda_1}\right)^{iT} B(\lambda_1)\overline{B(\lambda_2)} e^{-\left(\frac{H_1}{2}\ln\frac{\lambda_2}{\lambda_1}\right)^2} \right|,$$

$B(\lambda) = \frac{(P/\lambda)^{ih}-1}{\ln(P/\lambda)}$.

To estimate the sum Σ_1, using the inequality $\ln P/\lambda > \varepsilon_1 \ln P$ and relations (10), we obtain

$$\Sigma_1 = \sum_{\lambda < P^{1-\varepsilon_1}} \frac{a^2(\lambda)}{\lambda \ln^2(P/\lambda)} < \frac{1}{\varepsilon_1^2 \ln^2 P} \sum_{\lambda < P} \frac{a^2(\lambda)}{\lambda} \ll \frac{1}{\varepsilon_1^2 \ln P \ln X}.$$

Hence,

$$J_1 \ll H(\Sigma_1 + W_1) \ll H\left(\frac{1}{\varepsilon_1^2 \ln P \ln X} + (A\varepsilon_1^{-1}\mathcal{L}^{-2} + \varepsilon_1^{-2}\mathcal{L}^{-2})T^{-0,9\varepsilon}\right).$$

The integral of J_2 can be evaluated similarly to the evaluation of the integral of I_2:

$$J_2 \ll H_1 h^2(\Sigma_2 + W_2) \ll H_1 h^2(\varepsilon_1 \ln P \ln^{-1} X + T^{-0,9\varepsilon}),$$

$$J \ll H(\varepsilon_1^{-2}\ln^{-1} P \ln^{-1} X + (A\varepsilon_1^{-1}\mathcal{L}^{-2} + \varepsilon_1^{-2}\mathcal{L}^{-2})T^{-0,9\varepsilon}) + H_1 h^2(\varepsilon_1 \ln P \ln^{-1} X + T^{-0,9\varepsilon}),$$

$$I_3 \ll J + Hh^2\mathcal{L}^{-10} \leqslant c_1 H_1(\varepsilon_1^{-2}\ln^{-1} P \ln^{-1} X + A\varepsilon_1^{-1}T^{-0,9\varepsilon}\mathcal{L}^{-2} +$$

$$+\varepsilon_1^{-2}T^{-0.9\varepsilon}\mathcal{L}^{-2} + \varepsilon_1 h^2 \ln P \ln^{-1} X + h^2 T^{-0,9\varepsilon} + h^2\mathcal{L}^{-10}).$$

Considering the values of X, h, and H_1, i.e., $X = T^{0.01\varepsilon}$, $h = A\ln^{-1} T = A\mathcal{L}^{-1}$, $H_1 = H + h$, we have

$$\varepsilon_1^{-2}\ln^{-1} P \ln^{-1} X \leqslant 4h^2 A^{-2}\varepsilon^{-1}\varepsilon_1^{-2}, \quad A\varepsilon_1^{-1}T^{-0,9\varepsilon}\mathcal{L}^{-2} = h^2 A^{-1}\varepsilon_1^{-1}T^{-0,9\varepsilon},$$

$$\varepsilon_1^{-2}T^{-0,9\varepsilon}\mathcal{L}^{-2} = h^2 A^{-2}\varepsilon_1^{-1}T^{-0,9\varepsilon}.$$

Hence,

$$I_3 \leqslant c_1 Hh^2(1 + A\mathcal{L}^{-2}H^{-1})(A^{-2}\varepsilon^{-1}\varepsilon_1^{-2} + \varepsilon_1\varepsilon^{-1} + (A^{-1}\varepsilon_1^{-1} + A^{-2}\varepsilon_1^{-2} + 1)T^{-0,9\varepsilon} + \mathcal{L}^{-10}).$$

Without restricting generality, we will assume that $1 + A\mathcal{L}^{-2}H^{-1} < 0{,}1$, so assuming $c_2 = c_1$, we find

$$I_3 \leqslant c_2 Hh^2(A^{-2}\varepsilon^{-1}\varepsilon_1^{-2} + \varepsilon_1\varepsilon^{-1} + (A^{-1}\varepsilon_1^{-1} + A^{-2}\varepsilon_1^{-2} + 1)T^{-0,9\varepsilon} + \mathcal{L}^{-10}).$$

Now let's take

$$A = \left((32c_2 + 32)\varepsilon^{-1}\right)^{1.5}, \quad \varepsilon_1 = (32c_2 + 32)^{-1}\varepsilon,$$

then

$$A^2\varepsilon\varepsilon_1^2 = 32c_2 + 32, \quad \varepsilon_1\varepsilon^{-1} = (32c_2 + 32)^{-1}, \quad A\varepsilon_1 = (32c_2 + 32)^{1/2}\varepsilon^{-0,5}.$$

Thus, for I_3, we find

$$I_3 \leqslant c_3 Hh^2, \; c_3 \leqslant \frac{1}{16} + \left[\left(\frac{\varepsilon}{32} + \frac{c_2^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}}{\sqrt{32}} + 1\right)T^{-0,9\varepsilon} + \mathcal{L}^{-10}\right].$$

Let's take $T_0 = T_0(\varepsilon) > 0$ so that the expression in the square bracket is less than 1/16, then

$$I_3 \leqslant \frac{1}{8}Hh^2.$$

Thus, from the estimates I_1, I_2 and I_3, for relation (2), we obtain

$$\sqrt{\mu(E) I_2} \geqslant I_1 - \sqrt{HI_3} \geqslant hH - h^2 + O\left(T^{1/6} X \ln T\right) - \frac{1}{2\sqrt{2}}Hh \geqslant \frac{1}{2}hH,$$

$$\mu(E) \geqslant \frac{1}{4}h^2H^2I_2^{-1} \geqslant \frac{1}{4}h^2H^2(c(\varepsilon)h^2H_1)^{-1} = c_3H, \quad c_3 = c_3(\varepsilon) > 0.$$

Let us divide the interval $(T, T+H)$ into intervals of the form $(nh, nh+h)$, where $n = \left[\frac{T}{h}\right], \left[\frac{T}{h}\right] + 1, \ldots, \left[\frac{T+H}{h}\right]$. At least $[c_3Hh^{-1}] - 2$ of them contain points t of E. But if the interval $(nh, nh+2h)$ contains the point t of E, then the interval $(t, t+h)$, and hence the interval $(nh, nh+2h)$ contains at least one odd-order zero of the function $\zeta(1/2 + it)$. Consequently, the zeros of odd order of the function $\zeta(1/2+it)$ on the interval $(T, T + H)$ are at least as large as

$$\frac{1}{2}([c_3Hh^{-1}] - 2) \geqslant c_4H\ln T, \quad c_4 > 0.$$

which is precisely what I needed to prove.

4 | Conclusion

In this work, we investigated the coercive properties and separability of a fourth-order differential operator in a weighted function space. Through the establishment of coercive inequalities, we derived sufficient conditions that ensure the separability of the operator under consideration. These results build upon and extend existing research on second-order and biharmonic operators, providing a broader theoretical foundation for the analysis of higher-order differential equations. The developed criteria contribute significantly to the theory of elliptic differential operators, particularly in weighted Hilbert spaces, where traditional techniques may not be directly applicable. The findings also highlight the importance of weight functions and function classes in determining operator behavior. This study opens the door for future research on nonlinear operators, matrix potentials, and boundary value problems in complex domains, thereby deepening our understanding of the interplay between operator theory and functional analysis.

Author Contribution

Conceptualization, Sh.Kh.; Methodology, Sh.Kh.; Software, Sh.Kh.; Validation, Sh.Kh.; formal analysis, Sh.Kh.; investigation, Sh.Kh.; resources, Sh.Kh.; data maintenance, Sh.Kh.; writing-creating the initial design, Sh.Kh.; writing-reviewing and editing. All authors have read and agreed to the published version of the manuscript.

Data availability

All data supporting the reported findings in this research paper are provided within the manuscript.

Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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