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A Novel Analytical Solutions for Systems of Fractional Differential Equations using the Conformable Fractional Laplace Transform Method

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Abstract

In this paper, the conformable fractional Laplace transform method for solving systems of fractional differential equations is introduced. Both linear homogeneous and linear nonhomogeneous fractional differential systems, have been considered utilizing the conformable definition of the fractional derivative. The found solutions are plotted in 2D, which also demonstrate how the solutions are close to each other. Additionally, the exact solution for each case is reached as the fractional order goes to 1. Furthermore, Several numerical examples are included to demonstrate the precision and effectiveness of the proposed technique.

Keywords: Systems of fractional differential equations, Conformable fractional derivative, Laplace transform, Fractional differential equations.

1|Introduction

Over the years, the subject of fractional calculus has been studied by many researchers. This is an ongoing process, and we can recognize that new techniques and mechanisms are emerging in this field of fractional

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calculus, new techniques and mechanisms show up, which in turn make it possible to find important challenging insights and unknown correlations between many areas of physics. Fractional derivatives, due to their non locality properties, have proven their ability to describe several phenomena related to memory and after effects. Such phenomena are commonplace in physical processes, biological structures, and cosmological issues. Since it is difficult to find explicit solutions to these fractional problems [4, 8, 11], alternative methods [13, 14], numerical and approximate techniques [2, 3, 7] must be used. Conformable fractional derivatives have been used by many scholars in various academic disciplines.

The application of the conformable fractional derivatives is considered to be a fundamental and highly beneficial methodology. Furthermore, they enhance our ability to represent the behavior exhibited by concrete entities. Additionally, it modifies important transforms like the Laplace, Sumudu, and Nature transforms, which enable them to be useful instruments for solving singular FDEs [12]. Abdeljawad in [1], expanded the idea and presented the conformable Laplace transform as a generalization of the Laplace transform, building upon the concept of the conformable fractional derivative. The Laplace transform [9, 10], is a powerful technique for solving various linear partial differential equations having considerable significance in various fields such as engineering and applied sciences.

Overall, the Laplace transform is a valuable tool for solving systems of fractional differential equations. It offers a systematic approach to transform the problem into a more tractable form and then recover the solution in the original time domain.

In this work, we implement the conformable fractional Laplace transform method (CFLTM) for solving systems of differential equations. Both linear homogeneous and linear nonhomogeneous fractional differential systems, have been considered. We discuss how to solve fractional homogenous and nonhomogeneous systems of fractional differential equations using CFLTM.

The paper's remaining sections are arranged as follows: In Section 2, we first briefly retrieve the key definitions and theorems pertaining to conformable fractional calculus. Then, in Section 3, we introduce a novel technique for applying the CFLTM to systems of FDEs within the conformable sense, accompanied by a description of the theories and definitions related to fractional Laplace transform theory. To efficiently solve the fractional systems in the conformable sense, we extend the effective analytic fractional conformable Laplace algorithm with particular numerical examples in Section 4. Finally, in Section 5, we provide a summary of the key findings and offer concluding remarks.

2|Preliminaries

Let us give some needed definitions and theorems that will be considered in this paper.

Definition 1. [5] *Let $f:[0,\infty)\to\mathbb{R}$. The conformable fractional derivative of f with respect to t of order α is defined as*

$$D_t^{\alpha}(f)(t)=\lim_{\epsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\epsilon},\quad \textit{for all } t>0,\ \alpha\in(0,1).$$

If f is α-differentiable in some $(0,a)$, $a>0$, and $\lim_{t\to 0^+} D_t^{\alpha}(f)(t)$ exists, then $D_t^{\alpha}(f)(0)=\lim_{t\to 0^+} D_t^{\alpha}(f)(t)$.

Definition 2. [9] *Let $0<\alpha\le 1$ and $f:[0,\infty)\to\mathbb{R}$ be a real valued function. Then the conformable fractional Laplace transform of order α is defined as*

$$L_{\alpha}[f(t)](s)=\int_0^{\infty}\exp(-s\frac{t^{\alpha}}{\alpha})f(t)d^{\alpha}t=F_{\alpha}(s), \tag{1}$$

one can easily see that L_{α} is linear.

Definition 3. [9] *If $K>0$ and $T>0$ exist, then if there is a function $f:[0,\infty)\to\mathbb{R}$, it is said to be of conformable exponential order m such that*

$$|f(t)|\le Ke^{m\frac{t^{\alpha}}{\alpha}},\qquad \textit{for all } t\ge T.$$

Theorem 1. [1] *Assuming $f : [0, \infty) \to \mathbb{R}$ is a continuous function, $f^{(\alpha)}$ is a piecewise continuous real valued function with $0 < \alpha \leq 1$. If f is of conformable exponential order m, then*

$$L_\alpha[D_t^\alpha f(t)](s) = sF_\alpha(s) - f(0), \quad s > m, \tag{2}$$

where, $F_\alpha(s) = L_\alpha[f(t)]$.

To obtain additional information regarding conformable Laplace transform (CLT), please refer to [1, 6, 9].

3|Systems of Fractional Differential Equations

The steps of implementing the CFLTM to solve the following fractional systems are presented in detail in this part.

Consider the following system of linear fractional differential equations :

$$\begin{cases} y_1^{(\alpha)}(t) = a_1 y_1(t) + a_2 y_2(t) + f_1(t), \\ y_2^{(\alpha)}(t) = b_1 y_1(t) + b_2 y_2(t) + f_2(t), \end{cases} \tag{3}$$

with conditions : $y_1(0) = c_1$; $y_2(0) = c_2$,

where $0 < \alpha \leqslant 1,\ t > 0,\ f_1,\ f_2$ are source terms, and $y^{(\alpha)}$ is the conformable derivative.

Using the Laplace transform on both sides of Eqs. (3), we obtain

$$\begin{cases} L_\alpha[y_1^{(\alpha)}(t)] = L_\alpha[a_1 y_1(t) + a_2 y_2(t) + f_1(t)], \\ L_\alpha[y_2^{(\alpha)}(t)] = L_\alpha[b_1 y_1(t) + b_2 y_2(t) + f_2(t)]. \end{cases} \tag{4}$$

The linearity property of the Laplace transform enables us to obtain

$$\begin{cases} sL_\alpha[y_1(t)] - y_1(0) = a_1 L_\alpha[y_1(t)] + a_2 L_\alpha[y_2(t)] + L_\alpha[f_1(t)], \\ sL_\alpha[y_2(t)] - y_2(0) = b_1 L_\alpha[y_1(t)] + b_2 L_\alpha[y_2(t)] + L_\alpha[f_2(t)]. \end{cases} \tag{5}$$

Let us write: $Y_1(s)$ for $L_\alpha[y_1(t)]$, $Y_2(s)$ for $L_\alpha[y_2(t)]$ and $F_1(s)$ for $L_\alpha[f_1(t)]$, thus

$$\begin{cases} (s - a_1)Y_1(s) - a_2 Y_2(s) = c_1 + F_1(s), \\ (s - b_2)Y_2(s) - b_1 Y_1(s) = c_2 + F_2(s). \end{cases} \tag{6}$$

These are two equations in two unknowns. So, Eqs. (6) becomes:

$$\begin{cases} Y_1(s) - \frac{a_2}{s-a_1} Y_2(s) = \frac{c_1}{s-a_1} + \frac{1}{s-a_1} F_1(s), & (*) \\ \frac{-b_1}{s-b_2} Y_1(s) + Y_2(s) = \frac{c_2}{s-b_2} + \frac{1}{s-b_2} F_2(s). & (**) \end{cases}$$

Now, multiply Eq. $(*)$ by $\frac{b_1}{s-b_2}$ and summing the result with Eq. $(**)$, we get :

$$\frac{(s-a_1)(s-b_2) - a_2 b_1}{(s-a_1)(s-b_2)} Y_2(s) = \frac{c_2(s-a_1) + c_1 b_1 + (s-a_1)F_2(s) + b_1 F_1(s)}{(s-a_1)(s-b_2)}.$$

Therefore,

$$Y_2(s) = \frac{c_1 b_1 + b_1 F_1(s) + (c_2 + F_2(s))(s-a_1)}{(s-a_1)(s-b_2) - a_2 b_1}. \tag{7}$$

Replace $Y_2(s)$ in Eq. (6), we obtain :

$$Y_1(s) = \frac{a_2}{s-a_1} \times \frac{c_1 b_1 + b_1 F_1(s) + (c_2 + F_2(s))(s-a_1)}{(s-a_1)(s-b_2) - a_2 b_1} + \frac{c_1}{s-a_1} + \frac{1}{s-a_1} F_1(s)$$

Thus,

$$Y_1(s) = \frac{c_1}{s-a_1} + \frac{a_2}{s-a_1}[\frac{c_1 b_1 + b_1 F_1(s) + (c_2 + F_2(s))(s-a_1)}{(s-a_1)(s-b_2) - a_2 b_1}] + \frac{F_1(s)}{s-a_1}. \tag{8}$$

Operating the inverse Laplace transform on both sides in Eq. (7) and (8), we obtain

$$\begin{cases} y_1(t) = L_\alpha^{-1}[Y_1(s)], \\ y_2(t) = L_\alpha^{-1}[Y_2(s)]. \end{cases}$$

Hence, we get the exact solutions for the above system.

4|Applications

In this section, we use the conformable fractional Laplace transform method as an application to solve the following two examples of fractional linear homogeneous and linear nonhomogeneous systems of order 2×2.

Consider the following linear homogeneous system of fractional differential equations given by

$$\begin{cases} y_1^{(\alpha)}(t) = 2y_1(t) + y_2(t), \qquad 0 < \alpha \leq 1 \\ y_2^{(\alpha)}(t) = y_1(t) + 2y_2(t), \end{cases} \tag{9}$$

with: $y_1(0) = 2$, $y_2(0) = 1$.

By following the above steps, we have

$$\begin{cases} L_\alpha[y_1^{(\alpha)}(t)] = 2L_\alpha[y_1(t)] + L_\alpha[y_2(t)], \\ L_\alpha[y_2^{(\alpha)}(t)] = L_\alpha[y_1(t)] + 2L_\alpha[y_2(t)]. \end{cases} \tag{10}$$

Then, it becomes

$$\begin{cases} Y_1(s) - \frac{1}{s-2}Y_2(s) = \frac{2}{s-2}, \\ \frac{-1}{s-2}Y_1(s) + Y_2(s) = \frac{1}{s-2}. \end{cases} \tag{11}$$

Which are two equations in two unknowns. So, solving Eqs. (11), we get :

$$\begin{cases} Y_1(s) = \frac{2}{s-2} + \frac{s}{(s-1)(s-2)(s-3)}, \\ Y_2(s) = \frac{s}{(s-1)(s-3)}. \end{cases} \tag{12}$$

Hence,

$$\begin{cases} Y_1(s) = \frac{1}{2(s-1)} + \frac{3}{2(s-3)}, \\ Y_2(s) = \frac{-1}{2(s-1)} + \frac{3}{2(s-3)}. \end{cases} \tag{13}$$

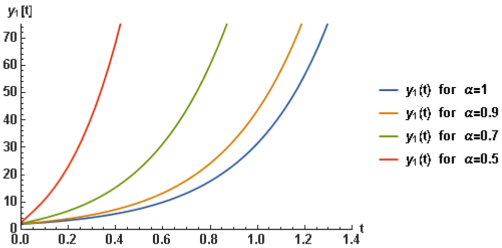
Therefore, the inverse Laplace transform for Eqs. (13) gives

$$\begin{cases} y_1(t) = L_\alpha^{-1}[Y_1(s)] = \frac{1}{2}e^{\frac{t^\alpha}{\alpha}} + \frac{3}{2}e^{\frac{3t^\alpha}{\alpha}}, \\ y_2(t) = L_\alpha^{-1}[Y_2(s)] = \frac{-1}{2}e^{\frac{t^\alpha}{\alpha}} + \frac{3}{2}e^{\frac{3t^\alpha}{\alpha}}, \end{cases} \tag{14}$$

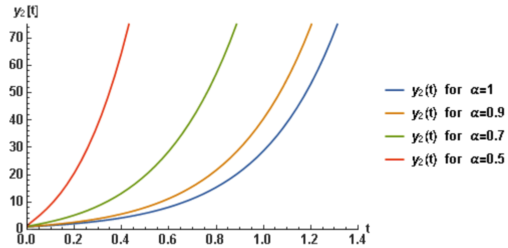
are solutions for the above system.

In fact, for $\alpha = 1$, the exact solutions of the classical system are

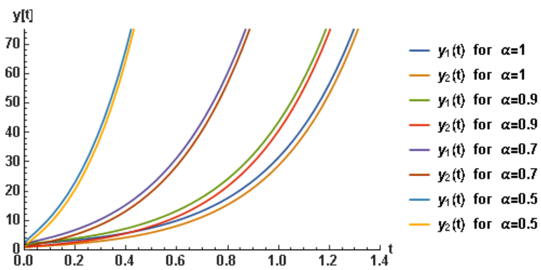
$$\begin{cases} y_1(t) = \frac{1}{2}e^t + \frac{3}{2}e^{3t}, \\ y_2(t) = \frac{-1}{2}e^t + \frac{3}{2}e^{3t}. \end{cases}$$



(A) Solution $y_1(t)$ in Conformable sense for different values of α.



(B) Solution $y_2(t)$ in Conformable sense for different values of α.



(C) Solution $y(t)$ in conformable sense for different values of α

FIGURE 1. The classical ($\alpha = 1$) and fractional solutions $y(t)$ for the system of equation (14) for $0 \leq t \leq 1.4$ and different values of α.

Now, consider the following linear nonhomogeneous system of fractional differential equations as follows :

$$\begin{cases} y_1^{(\alpha)}(t) = y_1(t) + y_2(t) + 2, & 0 < \alpha \leq 1 \\ y_2^{(\alpha)}(t) = -y_1(t) + y_2(t) - 2, \end{cases} \tag{15}$$

with: $y_1(0) = -2$, $y_2(0) = 1$.

By following the above steps we obtain :

$$\begin{cases} L_\alpha[y_1^{(\alpha)}(t)] = L_\alpha[y_1(t)] + L_\alpha[y_2(t)] + L_\alpha[2], \\ L_\alpha[y_2^{(\alpha)}(t)] = -L_\alpha[y_1(t)] + L_\alpha[y_2(t)] - L_\alpha[2]. \end{cases} \tag{16}$$

Hence,

$$\begin{cases} Y_1(s) = \frac{-2}{s} + \frac{1}{(s-1)^2+1}, \\ Y_2(s) = \frac{s-1}{(s-1)^2+1}. \end{cases} \tag{17}$$

Thus,

$$\begin{cases} y_1(t) = L_\alpha^{-1}[Y_1(s)] = -2 + e^{\frac{t^\alpha}{\alpha}} \sin(\frac{t^\alpha}{\alpha}), \\ y_2(t) = L_\alpha^{-1}[Y_2(s)] = e^{\frac{t^\alpha}{\alpha}} \cos(\frac{t^\alpha}{\alpha}), \end{cases} \tag{18}$$

are solutions for the given system.

In fact, for $\alpha = 1$, the exact solutions for the classical system are :

$$\begin{cases} y_1(t) = e^t \sin(t) - 2, \\ y_2(t) = e^t \cos(t). \end{cases}$$

(A) Solution $y_1(t)$ in Conformable sense for different values of α.

(B) Solution $y_2(t)$ in Conformable sense for different values of α.

FIGURE 2. The classical ($\alpha = 1$) and fractional solutions $y(t)$ for the system of equation (18) for $0 \le t \le 3.1$ and different values of α.

5|Conclusion

In the present paper the CFLTM has been successfully applied to compute exact solutions for systems of fractional differential equations, with examples of both homogeneous and nonhomogeneous linear differential equations. The found solutions are compared graphically in 2D for different values of α. These figures also demonstrate that the analytical solutions go to the exact one as $\alpha \to 1$. CFLTM can obtain a very accurate solution in only a few iterations. Therefore, we can conclude that the CFLTM method is very powerful and efficient in obtaining exact as well as numerical solutions.

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Author Contribution

Ilhem Kadri: methodology, software, editing, conceptualization, and writing. The author has read and agreed to the published version of the manuscript.

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Conflicts of Interest

The author declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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