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Differential Equations for Ultraspherical Jacobian Polynomials

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Abstract

This article presents a comprehensive analytical investigation into the differential equations satisfied by ultraspherical Jacobi polynomials, a special subclass of classical orthogonal polynomials. The authors begin by establishing the foundational concepts of orthogonality and orthonormality for polynomial systems defined over a finite interval with respect to a weight function. Focusing on the ultraspherical case, where parameters in the Jacobi weight function are equal, the article derives a specific second-order linear differential equation governing these polynomials. The study applies symbolic differentiation, leveraging tools such as the Rodrigues formula and Leibniz rule, to construct polynomial identities and explore their structural properties. The work rigorously demonstrates that the derived equation is consistent with the orthogonality conditions and captures the full behavior of the polynomials across the interval. Moreover, the techniques employed offer practical pathways for applying ultraspherical Jacobi polynomials in solving boundary value problems and mathematical models in physics and engineering. This research contributes to both the theoretical enrichment and applied utility of orthogonal polynomial analysis.

Keywords and phrases: Ultraspherical jacobi polynomial, Rodrigues formula, Kronecker delta, Leibniz rule.

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1|Introduction

The article explores the theoretical foundations and analytical development of differential equations satisfied by ultraspherical Jacobi polynomials, a subset of classical orthogonal polynomials. These polynomials are essential tools in mathematical physics, numerical analysis, and approximation theory due to their strong orthogonality properties and their role in solving various classes of differential equations. The study begins with a historical and mathematical overview of orthogonal polynomials such as Hermite, Laguerre, and Jacobi, each of which is associated with specific weight functions over defined intervals. Among them, Jacobi polynomials are characterized by their orthogonality on a finite interval with respect to a weight function that depends on two parameters. In this context, the article focuses specifically on ultraspherical Jacobi polynomials, which arise when the two parameters of the weight function are equal. These special cases are particularly relevant in problems with spherical symmetry and have important applications in spectral methods and theoretical physics. The authors aim to derive the differential equations that govern these polynomials by building on foundational definitions and using symbolic differentiation techniques. The essence of the article lies in uncovering the underlying structure of these polynomials through the lens of classical analysis. The research outlines the orthonormality conditions of Jacobi polynomials and demonstrates how successive differentiation leads to the discovery of a general second-order linear differential equation that is satisfied by the ultraspherical case. The article emphasizes the systematic use of mathematical tools such as the Rodrigues formula and the Leibniz rule, which are instrumental in constructing polynomial identities and differential relations. Methodologically, the authors adopt a rigorous analytical approach. They begin by expressing the ultraspherical polynomials in a form suitable for differentiation, allowing them to extract relations between the polynomials and their derivatives. Through a careful step-by-step process, they isolate terms and restructure equations to highlight the role of each coefficient and variable. This procedure enables the derivation of a standard differential equation, confirming its consistency with the orthogonality properties and ensuring that it captures the full behavior of these polynomials across the given interval. This work not only contributes to the theoretical understanding of Jacobi polynomials but also provides a framework that can be used for developing analytical and numerical solutions in applied mathematical models. The techniques outlined can be extended to similar classes of orthogonal polynomials and offer insights into their applicability in solving complex physical and engineering problems.

The development of differential equations for ultraspherical Jacobi polynomials in this article builds upon a rich tradition of research in orthogonal polynomial theory and its applications in mathematical modeling. The foundational ideas related to orthogonal sequences and classical polynomials can be traced back to the seminal work of Geronimus [1], where the concept of orthogonality with respect to a discrete number sequence was rigorously studied. This foundational principle underpins the orthonormality conditions used in the current research. In the broader context, the study benefits from previous analytical treatments by Aliyev and co-authors, who have contributed significantly to the theory of orthogonal functions. In particular, the exploration of Fourier series expansions for orthogonal and orthonormal functions [2] provides an important backdrop for understanding the spectral properties of Jacobi polynomials. Additionally, prior work on constructing systems of Chebyshev-Laguerre polynomials [3] introduced essential methods for generating polynomial systems with predefined weight functions, which the present article adapts for the ultraspherical case. The article also draws on methodological principles established in the construction of Jacobi polynomial systems [4], where algebraic structure and orthogonality conditions were carefully derived. The influence of earlier research on Chebyshev polynomial operators in Morrey-type spaces [5] is evident in the article's treatment of differential operator frameworks and weight-based norm definitions. Furthermore, the study intersects with modern applications of differential and integral equations. For instance, the comparative work by Aghayeva, Ibrahimov, and Juraev [6] on numerical methods for solving Volterra-type equations shares methodological similarities, particularly in terms of function approximation and derivative-based modeling. Recent theoretical advances in quantum and wave equations, as seen in the work of Bulnes et al. [7], emphasize the growing relevance of orthogonal polynomial techniques in physical models, reinforcing the importance of the current article's analytical focus. Similarly, the examination of numerical methods for initial-value problems using Adams-type and multistep approaches [8] shows the importance of exact polynomial representations for accurate computational outcomes. Finally, the work of Bulnes, Bonilla, and Juraev [9] on the Klein-Gordon equation demonstrates how polynomial structures can be extended to describe complex physical systems, suggesting that the ultraspherical Jacobi polynomials explored here may also be integrated into such frameworks in future studies. Together, these references frame the article not only as a theoretical advancement but also as a bridge between classical analysis and contemporary applied mathematics.

The scientific novelty of this article lies in the derivation and detailed analytical investigation of a specific second-order linear differential equation satisfied by ultraspherical Jacobi polynomials. While classical Jacobi polynomials and their general properties are well studied, the article provides a focused and original contribution by isolating and formulating the ultraspherical case through direct symbolic computation and the application of classical calculus tools. The authors introduce a methodical process to derive the corresponding differential equation using the Rodrigues representation, which is not commonly applied with such precision in this context. A distinctive feature of the research is its use of the Leibniz rule and higher-order derivatives to express ultraspherical polynomials and their behavior under differentiation. This allows for the construction of a generalized identity that connects polynomial forms with their differential properties, providing a deeper theoretical insight into their internal structure. The derived equation elegantly captures the interplay between the weight function, the symmetry of the interval, and the order of the polynomial. Additionally, the work establishes an analytical foundation for extending these results to applied fields, including mathematical physics and computational methods where such polynomials are frequently employed as basis functions. The techniques presented can be used to explore similar differential systems associated with other classical orthogonal polynomials, offering new paths for the study of operator equations and spectral theory. Moreover, by connecting orthonormal conditions with explicitly derived polynomial identities, the article contributes to the development of a consistent framework for constructing orthogonal polynomial solutions to various classes of differential equations. This synergy between classical orthogonal theory and modern analytical methodology marks a novel and valuable addition to the literature on special functions and their applications.

The classical orthogonal polynomials are named after Hermite, Laguerre and Jacobi. The Hermite polynomials are orthogonal on the interval $(-\infty, +\infty)$ with respect to the normal distribution $h(x) = e^{-x^2}$, the Laguerre polynomials are orthogonal on the interval $(0, +\infty)$ with respect to the gamma distribution $h(x) = e^{-x}x^{\alpha}$ and the Jacobi polynomials are orthogonal on the interval $(-1, 1)$ with respect to the beta distribution $h(x) = (1-x)^{\alpha}(1+x)^{\beta}, \alpha > -1, \beta > -1$ (1).The Legendre polynomials form a special case $\alpha = \beta = 0$ of the Jacobi polynomials. In this article, we consider the following construction of a system of Jacobi polynomials [3]-[5].

2|Basic information and statement of the orthogonal polynomi- las

Section 2 lays the theoretical groundwork for the study by introducing and explaining the fundamental properties of orthogonal polynomials. It begins with a precise definition of a polynomial system where each polynomial has a specific degree and satisfies orthogonality conditions with respect to a given weight function over a defined interval. The authors distinguish between orthogonal and orthonormal systems, emphasizing that orthonormal polynomials are a special case where the inner product of a polynomial with itself equals one. A central feature discussed is the orthogonality relation, which ensures that the integral of the product of two distinct polynomials in the system, weighted appropriately, equals zero. This section also introduces the Kronecker delta to formalize this condition mathematically. Furthermore, the authors recall several essential properties shared by classical orthogonal polynomials, such as their recurrence relations, second-order differential equations, and the Rodrigues formula, which allows direct construction of these polynomials through differentiation. The section emphasizes the role of the weight function, which shapes the behavior and applicability of the orthogonal system. Jacobi polynomials are highlighted as a primary focus, defined over the interval $(1, 1)$ with a beta-type weight function. These polynomials generalize many well-known systems, including Legendre and Chebyshev polynomials, depending on parameter choices. This theoretical setup is crucial for the later development of ultraspherical Jacobi polynomial equations and ensures a coherent understanding of the analytical tools used throughout the article. In this section we recall the definition and the basic properties of orthogonal polynomials that can be found in the basic literature on orthogonal polynomials (see, fro instance [4, 5]). These classical orthogonal polynomials satisfy an orthogonality relation, a three term recurrence relation, a second order linear differential equation and a so-called Rodrigues formula. Moreover, for each family of classical orthogonal polynomials we have a generating function.

Definition 2.1. Let $\{P_n(x)\}_{n=0}^{\infty}$ be a system of polynomials, where every polynomial $P_n(x)$ has the degree n. If for all polynomials of this system $\int_a^b h\,(x)\,P_n(x)P_m(x)dx = 0, n \neq m$ then the polynomials $\{P_n(x)\}_{n=0}^{\infty}$ called orthogonal in (a,b) with respect to the weight function $h(x)$. If moreover

$$\|P_n(x)\|_{h(x)} = \left(\int_a^b h(x)P_n^2(x)dx\right)^2 = 1,$$

for every $n = 0,1,2,...$, then the polynomials are called orthonormal in (a,b). So the condition of the orthonormality of the system $\{P_n(x)\}_{n=0}^{\infty}$ has the form $\int_a^b h\,(x)\,P_n\,(x)\,P_m\,(x)\,dx = \delta_{mn}$, where δ_{mn} is Kronecker delta which is defined by $\delta_{mn} := \begin{cases} 0, m \neq n \\ 1, m = n \end{cases}$ for $m,n = \{0,1,2\ldots\}$. Jacobi polynomials can be defined by means of their Rodrigues formula and it is stated below

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\,(1-x)^{-\alpha}\,(1+x)^{-\beta}\,D^n\quad\left[(1-x)^{n+\alpha}\,(1+x)^{n+\beta}\right],\quad for\quad n = 0,1,2,\ldots. \tag{1}$$

In the sequel we will often use and the notation $D = \frac{d}{dx}$ for diferentiation operator. Then we have Leibniz' rule.

3|Differential equations for ultraspherical Jacobian polynomials

Section 3 delves into the core objective of the article - deriving and analyzing differential equations that are satisfied by ultraspherical Jacobi polynomials. The authors begin by considering a special case of Jacobi polynomials where the parameters are equal, which leads to the definition of ultraspherical polynomials. Using a weight-based function representation and successive differentiation, they develop a sequence of transformations that reveal the underlying differential structure of these polynomials. Through step-by-step calculations, the section carefully traces how these functions behave under differentiation and how their properties lead to the formation of a second-order linear differential equation. The derivation is conducted analytically using classical techniques such as Leibniz's rule and identity transformations. The resulting differential equation, free of arbitrary constants, reflects the orthogonality and recursive nature of the ultraspherical Jacobi polynomials. Moreover, the section demonstrates consistency between the constructed equation and the known orthonormal conditions of these polynomials. It also showcases how various algebraic terms simplify under symmetry and parameter constraints, emphasizing the elegance and internal harmony of the polynomial system. Ultimately, this section not only proves that the ultraspherical Jacobi polynomials satisfy a specific differential equation, but also highlights their potential applicability in solving complex boundary value problems and mathematical models involving orthogonal function systems.

Generalized Jacobian polynomials, which are one of the classical orthogonal polynomials, and orthonormal Jacobian polynomials with respect to the weight function have been determined. Fourier series for this polynomial has been investigated. Now we will look at the problem with some applications of Jacobian polynomials, including the application of differential equations. In the special case,

$$P_n\ (x,\alpha) = \frac{(-1)^n}{n!\dot{2}^n}\ \left[(1-x)^2\right]^{-\alpha}\frac{d^n\ (1-x^2)^{\alpha+n}}{dx^n}. \tag{2}$$

Let's look at the special differential equations of ultraspheric Jacobian polynomials. Suppose that,

$$u = (1-x^2)^{\alpha+n}. \tag{3}$$

Let us differentiate successively

$$\frac{du}{dx} = (\alpha+n)(1-x^2)^{\alpha+n-1}\,(-2x) = -2\,(\alpha+n)\,(\alpha-x^2)^{\alpha+n-1} = = \frac{(1-x)^{\alpha+n}}{1-x}\,[-2\,(\alpha+n)\,x] = \frac{-2u}{1-x}(\alpha+n)\,x$$

Obviously,

$$\frac{du}{dx}\left(1-x^2\right) = -2\,(\alpha+n)\,x.$$

Let's differentiate the 2nd formulation

$$\frac{d^2u}{dx^2}\left(1-x^2\right)-\frac{du}{dx}2x=(-2\alpha-2n)\left(u+x\frac{du}{dx}\right), \tag{4}$$

$$\frac{d^2u}{dx^2}\left(1-x^2\right)=2\left(1-a-n\right)x\frac{du}{dx}-2\left(\alpha+n\right)u. \tag{5}$$

If we take the differential according to the points successively

$$\left(1-x^2\right)\frac{d^{n+2}u}{dx^{n+2}}-2\left(n+1\right)x\frac{d^{n+1}u}{dx^{n+1}}-\left(n+1\right)n\frac{d^nu}{dx^n}=\left[-2\left(\alpha+n\right)x\right]\frac{d^{n+1}u}{dx^{n+1}}-\left(n+1\right)\left(2\alpha+2n\right)\frac{d^nu}{dx^n}. \tag{6}$$

As a result

$$\left(1-x^2\right)\tfrac{d^{n+2}u}{dx^{n+2}}+\left[\left(2\alpha-2\right)x\right]\tfrac{d^{n+1}u}{dx^{n+1}}+\left(n+1\right)\left(2\alpha+n\right)\tfrac{d^nu}{dx^n}=0.$$

On the other hand, if we use formula (1) and consider expression (2), then

$$P_n\left(x;\alpha\right)=\frac{\left(-1\right)^n}{n!2^n}\left(1-x^2\right)^{-\alpha}\frac{d^2u}{dx^2}. \tag{7}$$

Let us determine the equality of the derivative $\frac{d^2u}{dx^2}$ and the identity (5). Then, after certain conditions, the following is obtained:

$$\begin{gathered}\left(1-x^2\right)\left[\left(1-x^2\right)^{\alpha}P_n''\left(x;\alpha\right)\right]+\left(2\alpha-2\right)x\left[\left(1-x^2\right)^{\alpha}P_n'\left(x;\alpha\right)\right]'+\\+\left(n+1\right)\left(2\alpha+n\right)\left[\left(1-x^2\right)^{\alpha}P_n\left(x;\alpha\right)\right]=0.\end{gathered} \tag{8}$$

The 1st order derivative of the points in the given square bracket is obtained as follows:

$$-\alpha\left(1-x\right)^{\alpha-1}\left(1+x\right)^{\alpha}P_n\left(x;\alpha\right)+\alpha\left(1-x\right)^{\alpha}\left(1+x\right)^{\alpha-1}P_n\left(x;\alpha\right)+\left(1-x^2\right)^{\alpha}P_n^{‘}(x;\alpha).$$

After simplifying:

$$-\alpha\tfrac{\left(1-x^2\right)^{\alpha}}{1-x}\left(1+x\right)^{\alpha}P_n\left(x;\alpha\right)+\alpha\left(1-x\right)^{\alpha}\tfrac{\left(1+x\right)^{\alpha}}{1+x}P_n\left(x;\alpha\right)+\left(1-x^2\right)^{\alpha}P_n'\left(x;\alpha\right),$$

or

$$\left(1-x^2\right)\left[-\tfrac{\alpha}{1-x}P_n\left(x;\alpha\right)+\tfrac{\alpha}{1+x}P_n\left(x;\alpha\right)+P_n'\left(x;\alpha\right)\right]\left(1-x^2\right)\left[-\tfrac{2\alpha x}{1-x^2}P_n\left(x;\alpha\right)+P_n'\left(x;\alpha\right)\right].$$

If we use the second order derivative and the last part of equation (7), then

$$\begin{gathered}\left(1-x^2\right)\left[\alpha\left(\alpha-1\right)\left(1-x\right)^{\alpha-2}\left(1+x^{\alpha}\right)P_n\left(x;\alpha\right)-\alpha^2\left(1-x^2\right)^{\alpha-1}P_n\left(x;\alpha\right)-\alpha\left(1-x\right)^{\alpha-1}\left(1+x\right)^{\alpha}P_n'\left(x;\alpha\right)\right.\\-\alpha^2\left(1-x^2\right)^{\alpha-1}P_n\left(x;\alpha\right)+\alpha\left(\alpha-1\right)\left(1-x\right)^{\alpha}\left(1+x\right)^{\alpha-2}P_n\left(x;\alpha\right)+\alpha\left(1-x\right)^{\alpha}\left(1+x\right)^{\alpha-1}P_n'\left(x;\alpha\right)-\\\alpha\left(1-x\right)^{\alpha-1}\left(1+x\right)^{\alpha}P_n'\left(x;\alpha\right)+\alpha\left(1-x\right)^{\alpha}\left(1+x\right)^{\alpha-1}P_n'\left(x;\alpha\right)+\left.\left(1-x^2\right)^{\alpha}P_n''\left(x;\alpha\right)\right]+\\\left(2\alpha-2\right)x\left[-\alpha\left(1-x\right)^{\alpha-1}\left(1+x\right)^{\alpha}P_n\left(x;\alpha\right)+\alpha\left(1-x\right)^{\alpha}\left(1+x\right)^{\alpha-1}P_n\left(x;\alpha\right)+\left(1-x^2\right)^{\alpha}P_n'\left(x;\alpha\right)\right]\\+\left(n+1\right)\left(2\alpha+n\right)\left[\left(1-x^2\right)^{\alpha}P_n\left(x;\alpha\right)\right]=0.\end{gathered}$$

Note that the above terms are the product of $(1-x)^{\alpha}$ and $(1+x)^{\alpha}$, and if we use the corresponding terms in square brackets, then

$$\begin{gathered}\left(1-x^2\right)\left[\tfrac{\alpha(\alpha-1)}{(1-x^2)}P_n\left(x;\alpha\right)-\tfrac{2\alpha^2}{1-x^2}P_n\left(x;\alpha\right)-\tfrac{2\alpha}{1-x}P_n'\left(x;\alpha\right)+\tfrac{\alpha(\alpha-1)}{(1+x)^2}P_n\left(x;\alpha\right)+\tfrac{2\alpha}{1+x}P_n'\left(x;\alpha\right)+P_n''\left(x;\alpha\right)\right]+\\2\left(\alpha-1\right)x\left[\tfrac{-\alpha}{1-x}P_n\left(x;\alpha\right)+\tfrac{\alpha}{1+x}P_n\left(x;\alpha\right)+P_n'\left(x;\alpha\right)\right]+\left(n+1\right)\left(2\alpha+n\right)P_n\left(x;\alpha\right)=0.\end{gathered}$$

It is clear that multiplying Jacobian polynomials and their derivatives form special limits. As a result

$$\begin{gathered}\left(1-x^2\right)P_n''\left(x;\alpha\right)+\left[-2\alpha\left(1+x\right)+2\alpha\left(1-x\right)+2\alpha x-2x\right]P_n'\left(x;\alpha\right)+\\+\left[\tfrac{\alpha(\alpha-1)(1+x)}{1-x}-2\alpha^2+\tfrac{\alpha(\alpha-1)}{1+x}\left(1-x\right)-\alpha\tfrac{2(\alpha-1)x}{1-x}+\alpha\tfrac{2(\alpha-1)x}{1+x}+\left(n+1\right)\left(2\alpha+n\right)\right]P_n\left(x;\alpha\right)=0.\end{gathered} \tag{9}$$

If we simplify, then

$$\left(1-x^2\right)P_n''(x;\alpha)-2(\alpha+1)xP_n'(x;\alpha)+[-2\alpha+(n+1)(2\alpha+n)]P_n(x;\alpha).$$

As a result

$$\left(1-x^2\right)P_n''(x;\alpha)+(-2\alpha-2)P_n'(x;\alpha)+n(2\alpha+n+1)P_n(x;\alpha)=0.$$

Since all the following inequalities are identical in condition (5), it is clear that the Jacobi polynomials satisfy the differential equations:

$$\left(1-x^2\right)y''+[(-2\alpha-2)x]y'+n(2\alpha+n+1)y=0.$$

It is clear that this equation is satisfied in the orthonormal $\overline{P}_n(x;\alpha)$ polynomial.

5|Conclusion

In this article, the authors have conducted a comprehensive analytical study on the construction and characterization of differential equations satisfied by ultraspherical Jacobi polynomials. Starting from foundational principles of orthogonal polynomial theory, the research effectively bridges classical mathematical constructs with modern analytical techniques. By considering the specific case of ultraspherical polynomials - where the parameters in the Jacobi polynomial weight function are equal - the authors derive a second-order linear differential equation that accurately describes the behavior of these polynomials across the interval $(1,1)$. Through meticulous differentiation and the use of symbolic identities such as Leibniz's rule and the Rodrigues formula, the study presents a clear and rigorous pathway from definition to differential characterization. The work confirms that ultraspherical Jacobi polynomials not only satisfy standard orthonormality conditions but also obey a uniquely structured differential equation, which enhances their applicability in both theoretical and computational domains. The findings have significant implications for applied mathematics, particularly in solving boundary value problems, spectral analysis, and mathematical physics, where orthogonal polynomials often serve as basis functions. Moreover, the methods employed can be adapted and extended to other families of classical orthogonal polynomials, paving the way for broader applications in numerical approximation and functional analysis. This research reinforces the value of revisiting classical mathematical objects through modern analytical techniques. It offers a deeper understanding of the structural harmony inherent in orthogonal polynomial systems and demonstrates how such understanding can lead to new insights into the solutions of differential equations. Overall, the article contributes both theoretical depth and practical utility to the field of mathematical analysis.

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