Karshi Multidisciplinary International Scientific Journal

www.kmisj.reapress.com

**K. Multidiscip. Int. Sci. J. Vol. 2, No. 1 (2025) 51–59.**

Paper Type: Original Article

# Taylor Expansions of $q-$Pochhammer Symbols via the Leverrier–Takeno Method

**Juan José Diaz Bulnes**[1,*], **Irfan Nurhidayat**[2], **Mahmood Ahmad Pathan**[3,4], **Hemant Kumar**[5], **José Luis López-Bonilla**[6]

[1]Department of Exact and Technological Sciences, Federal University of Amapá, Rod. J. Kubitschek, 68903-419, Macapá, AP, Brazil; bulnes@unifap.br.

[2]Postdoctoral Department, International Engineering and Technology Institute, Hong Kong, China; irfannurhidayat09@gmail.com.

[3]Centre for Mathematical and Statistical Sciences, Peechi Campus, Peechi, Kerala, India-680 653; mapathan@gmail.com.

[4]Department of Mathematics, Aligarh Muslim University, Aligarh, India-202 002; mapathan@gmail.com.

[5]Department of Mathematics, D. A-V. Postgraduate College, Kanpur, Uttar Pradesh, India-208 001; hemantkumar_kn03@csjmu.ac.in.

[6]ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México; jlopezb@ipn.mx.

**Citation:**

| | |
|---|---|
| *Received: 14 October 2024*<br>*Revised: 21 December 2024*<br>*Accepted: 30 January 2025* | Bulnes, J. D., Nurhidayat, I., Pathan, M. A., Kumar, H., & López-Bonilla, J. (2025). Taylor expansions of $q-$ pochhammer symbols via the leverrier–takeno method. *Karshi multidisciplinary international scientific journal*, 2(1), 51-59. |

## Abstract

This article presents a new approach to the study of generalized $q-$Pochhammer symbols through the Leverrier–Takeno method. By linking the derivatives of these products to the coefficients of characteristic polynomials, the work provides a systematic framework for constructing Taylor expansions in terms of combinatorial objects such as Bell polynomials and $q-$binomial coefficients. The method naturally leads to several well-known results in $q-$series, including the $q-$binomial theorem, Euler's identity, and the Jacobi triple product identity. In addition, the approach reveals close connections with partition functions and recurrence relations, offering new perspectives on classical number theoretic results. The findings highlight the effectiveness of combining linear algebraic methods with combinatorial techniques to deepen the understanding of $q-$series and their applications in mathematics.

**Keywords:** Characteristic polynomial, $q-$binomial theorem, Leverrier-Takeno's procedure, Taylor expansion, Complete Bell polynomials, Partition function, Newton's recurrence expression, $q-$series, Jacobi triple product identity.

Corresponding Author: bulnes@unifap.br

https://doi.org/10.22105/kmisj.v2i1.88

# 1|Introduction

The theory of $q-$series plays a central role in modern mathematics, with applications ranging from combinatorics and number theory to mathematical physics. Among its fundamental objects are q-Pochhammer symbols, which appear in a variety of identities and expansions, including the celebrated $q-$binomial theorem and the Jacobi triple product identity. These identities are not only elegant in their own right but also provide important connections to partition functions and recurrence relations. In this paper, we propose a novel approach to the study of $q-$Pochhammer symbols by employing the Leverrier–Takeno method, a classical technique from matrix theory. This method enables us to express derivatives and expansions in terms of characteristic polynomials and Bell polynomials, thereby establishing new links between linear algebra, combinatorics, and q-series. In this article, we developed a systematic procedure to obtain Taylor expansions of generalized $q-$Pochhammer symbols through the Leverrier–Takeno technique. The approach revealed deep connections between the coefficients of characteristic polynomials, Bell polynomials, and q-binomial coefficients, offering a unified framework for deriving classical results of q-series. Our analysis confirmed the appearance of well-known identities such as the q-binomial theorem, Euler's identity, and the Jacobi triple product identity, while also showing their implications for partition functions and recurrence relations. The results highlight the effectiveness of matrix methods in addressing problems of number theory and combinatorics, providing a bridge between different areas of mathematics.

The present article situates itself within the long tradition of research on $q-$series and related combinatorial structures. The foundations of this field go back to the pioneering works of Jacobi [3] , who introduced the theory of elliptic functions and formulated the famous triple product identity. Later developments have been systematically presented in modern references such as Chan [1], Johnson [2], and Hirschhorn [4], which emphasize the deep connections between $q-$series, partition functions, and combinatorial identities. The $q-$binomial theorem and its various consequences remain central to this domain. Landmarks in understanding such expansions have been explored through matrix analysis [5], [6] and more recently by López-Bonilla and collaborators [7], who related characteristic polynomials to Bell polynomials. These connections motivate the approach taken in this article, where the Leverrier–Takeno method is applied to study expansions of $q-$Pochhammer symbols. The Leverrier–Takeno technique itself has a long history, starting with the works of Leverrier [8] in celestial mechanics and later generalized by Takeno [9] for tensors and matrices. Its applications to physical and mathematical problems were further elaborated by Wilson et al. [10] and by Guerrero-Moreno and colleagues [12]. More recent contributions highlight refinements and alternative formulations, as in Bulnes, Islam, and López-Bonilla [11]. An essential tool in our analysis is the use of complete Bell polynomials, which appear in classical combinatorial texts such as Riordan [16] and Comtet [17], and in more modern treatments like Connon [18]. Several articles by López-Bonilla and coauthors [19]-[22] have deepened the applications of these polynomials to problems in algebra and analysis. Their link to Newton's recurrence relations [23], as well as to triangular matrix inversion [25], provides a structural basis for the method employed here. Parallel developments in linear algebra and matrix theory also contribute to the methodology. Csanky [24] proposed efficient algorithms for matrix inversion, while Wang, Wei, and Qiao [26] studied generalized inverses. Classical Russian contributions, such as those of Faddeev and Sominsky [27], [29] laid the groundwork for algorithmic approaches, later refined by Gower [30] and Helmberg and collaborators [31]. These studies confirm the robustness of the Leverrier–Faddeev type algorithms, as also emphasized by Hou [32], [33] and by López-Bonilla and his collaborators in applications to spacetime models [34]. From the perspective of number theory, the article engages with results concerning partition functions. Malenfant [40] derived finite expressions for partition numbers, later revisited by López-Bonilla and Vidal-Beltrán [41], and by Bulnes and coauthors [42]. Hirschhorn [44] and Berkovich and Uncu [43] further explored congruences and polynomial identities that echo the results obtained here. The relevance of partition identities to combinatorics and recurrence sequences was also emphasized by Birmajer et al. [46]. The determinant formulations of Vein and Dale [47], together with Gould's combinatorial identities [48], provide a further algebraic background to the type of expansions analyzed in this paper. These references underscore the long-standing interplay between determinants, polynomial identities, and $q-$series. In summary, this article builds on the foundational works of Jacobi [3] and Ramanujan-inspired studies, while drawing on modern expositions of q-series [1], [2], [4]. It combines the algebraic insights of Leverrier–Takeno methods [8], [9], [11], [12] with the combinatorial depth of Bell polynomials [16]-[21]. By linking these techniques to partition theory [40]-[42], the paper contributes to a deeper understanding of $q-$Pochhammer expansions and their applications.

Here we study the $q-$Pochhammer symbol:

$$A := (-z\ g(q); f(q))_n = (1 + zg)(1 + zgf)(1 + zgf^2)\ ...\ (1 + zgf^{n-1}), \tag{1}$$

which is a polynomial in $z$ of degree $n$, and its Taylor expansion is given by:

$$A = \sum_{k=0}^{n} \left[\frac{d^k A}{dz^k}\right]_{z=0} \frac{z^k}{k!}, \tag{2}$$

where usually $g(q)$ and $f(q)$ are powers of $q$. We shall have special interest in the case $g = -1$, $f = q$ because $(z; q)_n$ appears in the important $q-$binomial theorem [1, 2], that is, our approach gives an alternative motivation / deduction of this theorem. Besides, it is attractive the case $g = q$, $f = q^2$ because $(-zq; q^2)_\infty$ participates in the valuable Jacobi triple product identity [1, 2, 3, 4].

In Sec. 3 we find that the $d^kA/dz^k$ are related with the coefficients of the characteristic polynomial $p(\lambda)$ [5, 6, 7] of the following diagonal matrix:

$$B_{n\times n} = Diag\Big( \frac{1}{1 + zg}, \frac{f}{1 + zgf}, \frac{f^2}{1 + zgf^2}, \ ... \ , \frac{f^{n-1}}{1 + zgf^{n-1}} \Big) \tag{3}$$

and the traces of its powers are given by:

$$s_m = tr\ B^m = \sum_{j=1}^{n} \frac{f^{m(j-1)}}{\left(1 + zgf^{j-1}\right)^m}, \tag{4}$$

which are employed by the Leverrier-Takeno method [8, 9, 10, 11, 12] to construct $p(\lambda)$ for $B$:

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ ... \ + a_{n-1}\lambda + a_n\ , \tag{5}$$

where the quantities $a_r$ are in terms of the traces $s_j$; then in Sec. 2 we indicate the principal expressions of this method. Finally, in Sec. 3 we obtain the result:

$$\frac{d^k A}{dz^k} = (-g)^k\ k!\ A\ a_k, \qquad 0 \le k \le n, \tag{6}$$

thus (2) implies the relation:

$$(-zg; f)_n = \sum_{k=0}^{n} [a_k]_{z=0}\ (-gz)^k, \tag{7}$$

because $[A]_{z=0} = 1$, and we find that:

$$[a_k]_{z=0} = (-1)^k\ f^{\ (k(k-1))/2}\ \Pi_{j=1}^{k} \frac{1 - f^{n-j+1}}{1 - f^j} = (-1)^k\ f^{\ (k(k-1))/2} \binom{n}{k}_f, \tag{8}$$

such that $\binom{n}{k}_f$ is a $q-$binomial coefficient (Gaussian polynomial) [1, 2]. Hence from (7) and (8):

$$(-zg; f)_n = \sum_{k=0}^{n} \binom{n}{k}_f f^{\ (k(k-1))/2}\ (gz)^k, \tag{9}$$

and in Sec. 4 we make applications of this expression, for example, from it are immediate the Euler identity [4] and a $q-$series connected with the partition function [2].

# 2|Leverrier-Takeno Technique

For any matrix $B_{n\times n}$, in the approach of Leverrier-Takeno [8, 9, 10, 11, 12] the $a_i$ are determined with the Newton's recurrence relation:

$$r\ a_r + s_1\ a_{r-1} + s_2\ a_{r-2} + \ ...\ + s_{r-1}\ a_1 + s_r = 0, \quad r = 1, 2,\ ...\ , n, \quad a_0 = 1, \tag{10}$$

therefore:

$$\begin{gathered} a_1 = -s_1, \quad 2!\ a_2 = (s_1)^2 - s_2, \quad 3!\ a_3 = -(s_1)^3 + 3s_1s_2 - 2s_3, \\ 4!\ a_4 = (s_1)^4 - 6(s_1)^2 s_2 + 8s_1s_3 + 3(s_2)^2 - 6s_4, \\ 5!\ a_5 = -24s_5 - (s_1)^5 + 10(s_1)^3 s_2 - 20(s_1)^2 s_3 - 15(s_2)^2 s_1 + 30 s_1 s_4 + 20 s_2 s_3,\ etc., \end{gathered} \tag{11}$$

in particular, $detB = (-1)^n a_n$, that is, the determinant of any matrix only depends on the traces $s_r$, which means that $B$ and its transpose have the same determinant. In [13, 14, 15] we find the general expression:

$$a_k = \frac{(-1)^k}{k!} \begin{vmatrix} s_1 & k-1 & 0 & ... & 0 \\ s_2 & s_1 & k-2 & ... & 0 \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & ... & . \\ s_{k-1} & s_{k-2} & ... & ... & 1 \\ s_k & s_{k-1} & ... & ... & s_1 \end{vmatrix}, \quad k = 1,\ ...\ , n, \tag{12}$$

which allows reproduce the values (11).

Besides, we can exhibit a relation to determine the coefficients $a_j$ via the complete Bell polynomials [7, 16, 17, 18, 19, 20, 21, 22], in fact, we know the following representation [7]:

$$m!\ a_m = B_m\Big( -0!\ s_1, -1!\ s_2, -2!\ s_3, -3!\ s_4,\ ...\ , -(m-2)!\ s_{m-1}, -(m-1)!\ s_m \Big). \tag{13}$$

such that [7, 23]:

$$B_m(x_1, x_2,\ ...\ , x_m) = \begin{vmatrix} \binom{m-1}{0}x_1 & \binom{m-1}{1}x_2 & ... & \binom{m-1}{m-2}x_{m-1} & \binom{m-1}{m-1}x_m \\ -1 & \binom{m-2}{0}x_1 & ... & \binom{m-2}{m-3}x_{m-2} & \binom{m-2}{m-2}x_{m-1} \\ 0 & -1 & ... & \binom{m-3}{m-4}x_{m-3} & \binom{m-3}{m-3}x_{m-2} \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & ... & . \\ 0 & 0 & ... & \binom{1}{0}x_1 & \binom{1}{1}x_2 \\ 0 & 0 & ... & -1 & \binom{0}{0}x_1 \end{vmatrix}, \tag{14}$$

therefore:

$$\begin{gathered} B_0 = 1, B_1 = x_1, B_2 = x_1^2 + x_2, B_3 = x_1^3 + 3x_1x_2 + x_3,\ B_4 = x_1^4 + 6x_1^2x_2 + 4x_1x_3 + 3x_2^2 + x_4, \\ B_5 = x_1^5 + 10\ x_1^3x_2 + 10\ x_1^2x_3 + 15\ x_1x_2^2 + 5\ x_1x_4 + 10\ x_2x_3 + x_5,\ ... \end{gathered} \tag{15}$$

We see that (13) and (15) imply (11) if we employ $x_1 = -s_1$, $x_2 = -s_2$, $x_3 = -2s_3$, $x_4 = -6s_4$, $x_5 = -24s_5$, ... ; thus the coefficients of the characteristic polynomial (5) are generated by the complete Bell polynomials.

In the Newton's formula (10) the quantities $s_r$ are known, and the $a_j$ are solutions of the triangular linear system [24, 25, 26]:

$$C_{n\times n} \cdot (a_j)_{n\times 1} = \begin{pmatrix} 1 & 0 & 0 & ... & ... & 0 \\ s_1 & 2 & 0 & ... & ... & 0 \\ s_2 & s_1 & 3 & ... & ... & . \\ . & . & s_1 & ... & ... & . \\ . & . & . & ... & ... & . \\ . & . & . & ... & ... & 0 \\ s_{n-1} & s_{n-2} & s_{n-3} & ... & s_1 & n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ . \\ . \\ . \\ a_n \end{pmatrix} = - \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ . \\ . \\ . \\ s_n \end{pmatrix}, \tag{16}$$

then:

$$\begin{pmatrix} a_1 \\ . \\ . \\ . \\ a_n \end{pmatrix} = -C^{-1} \begin{pmatrix} s_1 \\ . \\ . \\ . \\ s_n \end{pmatrix}, \tag{17}$$

which gives the opportunity to invert a triangular matrix via interesting algorithms applying the Faddeev−Sominsky method [11, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36], matrix multiplication [37, 38] or binomial series [39].

# 3|Calculation of $\frac{d^k}{dz^k}(-zg; f)_n$

Here we calculate the derivatives of the product (1). From (4) it is useful to observe that:

$$\frac{d}{dz} s_m = -mg\ s_{m+1}; \tag{18}$$

then:

$$\frac{dA}{dz} = gA \sum_{j=1}^{n} \frac{f^{j-1}}{1+zgf^{j-1}} = gA\ trB = gAs_1 = -a_1, \quad \frac{d^2A}{dz^2} = g^2 A(s_1^2 - s_2) = 2!\ g^2 A\ a_2,$$
$$\frac{d^3A}{dz^3} = g^3 A(s_1^3 - 3s_1s_2 + 2s_3) = -3!\ g^3 A\ a_3, \tag{19}$$
$$\frac{d^4A}{dz^4} = g^4 A(s_1^4 - 6s_1^2 s_2 + 3s_2^2 + 8s_1s_3 - 6s_4) = 4!\ g^4\ A\ a_4,\ \ldots,$$

which is the result (6), that is, the derivatives of (1) are determined by the coefficients of the characteristic polynomial of the matrix (3). From (4):

$$[s_m]_{z=0} = \frac{1-f^{mn}}{1-f^m}, \tag{20}$$

whose application in (11) gives the expressions:

$$[a_1]_{z=0} = -\frac{1-f^n}{1-f},$$
$$[a_2]_{z=0} = f\ \frac{(1-f^n)(1-f^{n-1})}{(1-f)(1-f^2)},\ \ [a_3]_{z=0} = -f^3\ \frac{(1-f^n)(1-f^{n-1})(1-f^{n-2})}{(1-f)(1-f^2)(1-f^3)}, \tag{21}$$
$$[a_4]_{z=0} = f^6\ \frac{(1-f^n)(1-f^{n-1})(1-f^{n-2})(1-f^{n-3})}{(1-f)(1-f^2)(1-f^3)(1-f^4)},\ \ldots$$

in harmony with the relation (8); then (9) is immediate from (7) and (8).

# 4|Applications of (9)

a). $f = q$, $g = -1$, in this case the relation (9) implies the $q-$binomial theorem [1, 2]:

$$(z; q)_n = \sum_{k=0}^{n} \binom{n}{k}_q q^{(k(k-1))/2}\ (-z)^k. \tag{22}$$

b). From (9) if $\lim_{t\to\infty} f^t = 0$:

$$(-zg; f)_\infty = \sum_{k=0}^{\infty} \frac{f^{k(k-1)/2}}{(f; f)_k}\ (gz)^k, \tag{23}$$

because we know that [2]:

$$\lim_{n\to\infty} \binom{n}{k}_f = \frac{1}{(f; f)_k}. \tag{24}$$

For the case $g = q$ with $f = q^2$, the result (23) generates the expansion:

$$(-zg;q^2)_\infty = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q^2;q^2)_k} z^k, \tag{25}$$

which is important by its presence in the Jacobi triple product identity [1, 2, 3, 4]:

$$\sum_{n=-\infty}^{\infty} q^{n^2}\ z^n = (q^2;q^2)_\infty(-zq;q^2)_\infty(-\frac{q}{z};q^2)_\infty, \tag{26}$$

this relation is one of the fundamental identities in $q-$series. In (26) we can use (25) to obtain the property:

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n(n+m)}}{(q;q)_n(q;q)_{n+m}}\ , \quad m = 0, 1, 2, 3,\ ... \tag{27}$$

where $p(n)$ is the partition function [2, 40, 41, 42].

If $g = 1$ with $f = q$, then (23) implies the Euler identity [4, 43]:

$$(-z;q)_\infty = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q;q)_k} z^k, \tag{28}$$

which for $z = -q$ gives the inverse of (27) [2, 41]:

$$\sum_{n=0}^{\infty} b_n q^n = (q;q)_\infty = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)/2}}{(q;q)_k}, \tag{29}$$

where:

$$b_j = \begin{cases} 0, & if\ \ j \neq \frac{N}{2}(3N+1), \\ & \qquad\qquad N\ =\ 0,\ \pm1,\ \pm2,\ ... \\ (-1)^N, & if\ \ j = \frac{N}{2}(3N+1) \end{cases} \tag{30}$$

From (27) and (29) [44, 45]:

$$\sum_{n=0}^{\infty} p(n)t^n = \frac{1}{\sum_{r=0}^{\infty} b_r t^r}, \quad p(0) = b_0 = 1, \tag{31}$$

with the recurrence relation [46]:

$$\sum_{k=0}^{n} b_k\ p(n-k) = 0, \tag{32}$$

discovered by MacMahon [45]. On the other hand, from [47, 48] we have that relations of type (31) are equivalent to the following Hessenberg determinant:

$$p(n) = (-1)^n \begin{vmatrix} b_1 & b_0 & 0 & 0 & ... & 0 \\ b_2 & b_1 & b_0 & 0 & ... & 0 \\ b_3 & b_2 & b_1 & b_0 & ... & 0 \\ . & . & . & . & ... & . \\ . & . & . & . & ... & . \\ . & . & . & . & ... & b_0 \\ b_n & b_{n-1} & b_{n-2} & ... & ... & b_1 \end{vmatrix}, \tag{33}$$

obtained by Malenfant [40, 41].

*Remark 1.*- We know the $q-$binomial series [1]:

$$\frac{1}{(z;q)_n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k}_q z^k, \tag{34}$$

where we can use (22) to deduce the interesting identity:

$$\sum_{j=0}^{k} (-1)^j \binom{n+k-j-1}{k-j}_q \binom{n}{j}_q q^{j(j-1)/2} = 0, \quad n \geq 0, \ \ k \geq 1, \tag{35}$$

and it gives the following expression if $n \to \infty$:

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j}_q q^{j(j-1)/2} = 0, \quad k \geq 1, \tag{36}$$

which is immediate from $q-$binomial theorem for $z = 1$. Besides, from (27) and (34):

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p(n) q^n = \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k} \; . \tag{37}$$

We consider that the Leverrier-Takeno method and the Chebyshev polynomials [45] are useful tools to study $q-$Pochhammer symbols.

# 7|Conclusion

In this work, we developed a comprehensive study of generalized $q-$Pochhammer symbols using the Leverrier–Takeno method. By establishing a connection between derivatives of these products and the coefficients of characteristic polynomials, we demonstrated how matrix techniques can be effectively applied to problems in combinatorics and number theory. The framework allowed us to employ complete Bell polynomials and Newton-type recursions to systematically derive expansions, thereby unifying algebraic and combinatorial approaches. Our analysis revealed that classical results in $q-$series, such as the $q-$binomial theorem, Euler's identity, and the Jacobi triple product identity, naturally emerge from this methodology. Furthermore, the study showed how these expansions are directly linked to partition functions and their recurrence relations, providing deeper insight into the structural properties of integer partitions. The recurrence schemes and determinant formulations that appear highlight the versatility of the approach and its potential applications beyond the examples considered here. The significance of these results lies not only in re-deriving known identities through an alternative path but also in demonstrating the usefulness of linear algebraic methods in fields traditionally dominated by analytic or combinatorial techniques. This interdisciplinary perspective underscores the potential for further research at the intersection of matrix theory, combinatorics, and analytic number theory. Future work may include the extension of this method to other special functions within $q-$analysis, exploring connections with generalized binomial coefficients, or investigating identities related to modular forms. Additionally, the approach may find applications in mathematical physics, particularly in problems where partition functions and generating functions play a central role. In conclusion, the article provides a new algebraic viewpoint for the study of $q-$Pochhammer symbols and related $q-$series identities. It opens avenues for both theoretical exploration and practical applications, reinforcing the deep unity between linear algebra, combinatorics, and number theory.

# Acknowledgments

The authors would like to express their sincere gratitude to the editors and anonymous reviewers for their invaluable comments and constructive feedback, which significantly contributed to the enhancement of this paper.

# Funding

The authors declare that no external funding or support was received for the research presented in this paper, including administrative, technical, or in-kind contributions.

# Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings.

# References


[1] Hei-Chi Chan (2011). *An invitation to q−series: From Jacobi's triple product identity to Ramanujan's "most beautiful identity"*. World Scientific, Singapore.https://books.google.com/books

[2] Johnson, W.P. (2020). *An Introduction to q−Analysis*. Am. Math. Soc., Providence, Rhode Island, USA.https://books.google.com/books

[3] Jacobi, C.G.J. (1829). *Fundamenta Nova Theoriae Functionum Ellipticarum*. Sumptibus Fratrum Bornträger, Regiomonti.https://books.google.com/books

[4] Hirschhorn, M.D. (2017). *The Power of q. A Personal Journey*. Springer, Switzerland.

[5] Horn, R.A. & Johnson, Ch. R. (2013). *Matrix Analysis*. Cambridge University Press.

[6] Lanczos, C. (1988). *Applied Analysis*. Dover, New York.https://books.google.com/books

[7] López-Bonilla, J., Vidal-Beltrán, S. & Zúñiga-Segundo, A. (2018). Characteristic equation of a matrix via Bell polynomials. *Asia Mathematika*, 2(2), 49–51.http://www.asiamath.org/issue1/AM-1904-3103.pdf

[8] Leverrier, U.J.J. (1840). Sur les variations séculaires des éléments elliptiques des sept planétes principales. *J. de Math. Pures Appl. Série (1)*, 5, 220–254.https://www.numdam.org/item/JMPA_1840_1_5__220_0.pdf

[9] Takeno, H. (1954) A theorem concerning the characteristic equation of the matrix of a tensor of the second order. *Tensor NS*, 3, 119–122.

[10] Wilson, E.B., Decius, J.C. & Cross, P.C. (1980). *Molecular Vibrations*. Dover, New York, 216–217.

[11] Bulnes, J.D., Islam, N. & López-Bonilla, J. (2022). Leverrier-Takeno and Faddeev-Sominsky algorithms. *Scientia Magna*, 17(1), 77–84.https://sig-cdn.unifap.br/arquivos/2023189017d2c367821401a682359756/Leverrier-Takeno_and_Faddeev-Sominsky_algorithms.pdf

[12] Guerrero-Moreno, I., López-Bonilla, J. & Rivera-Rebolledo, J. (2011). Leverrier-Takeno coefficients for the characteristic polynomial of a matrix. *J. Inst. Eng. (Nepal)*, 8(1-2), 255–258.https://nepjol.info/index.php/JIE/article/download/5118/4254

[13] Brown, L.S. (1994). *Quantum Field Theory*. Cambridge University Press.https://books.google.com/books

[14] Curtright, T.L. & Fairlie, D.B. (2012). *A Galileon Primer*. arXiv: 1212.6972v1 [hep-th] 31 Dec.https://doi.org/10.48550/arXiv.1212.6972

[15] López-Bonilla, J., López-Vázquez, R. & Vidal-Beltrán, S. (2018) . An alternative to Gower's inverse matrix. *World Scientific News*, 102, 166–172.https://agro.icm.edu.pl/agro/element/bwmeta1.element.agro-be0117c6-c40e-454d-bafa-76a3db0e80f0

[16] Riordan, J. (1968). *Combinatorial Identities*. John Wiley and Sons, New York.

[17] Comtet, L. (1974). *Advanced Combinatorics*. D. Reidel Pub., Dordrecht, Holland.

[18] Connon, D.F. (2010). *Various Applications of the (Exponential) Complete Bell Polynomials*. http://arxiv.org/ftp/arxiv/papers/1001/1001.2835.pdf 16 Jan.

[19] López-Bonilla, J., López-Vázquez, R. & Vidal-Beltrán, S. (2018). Bell polynomials. *Prespacetime J.*, 9(5), 451–453.

[20] López-Bonilla, J., Vidal-Beltrán, S. & Zúñiga-Segundo, A. (2018). On certain results of Chen and Chu about Bell polynomials. *Prespacetime J.*, 9(7), 584–587.

[21] López-Bonilla, J., Vidal-Beltrán, S. & Zúñiga-Segundo, A. (2018). Some applications of complete Bell polynomials. *World Eng. & Appl. Sci. J.*, 9(3), 89–92.

[22] Pathan, M.A., Kumar, H., López-Bonilla, J. & Sherzad Taher, H. (2023). Identities involving generalized Bernoulli numbers and partial Bell polynomials with their applications. *Jnanabha*, 53(1), 272–276.

[23] Johnson, W.P. (2002). The curious history of Faà di Bruno's formula. *The Math. Assoc. of America*, 109, 217–234.https://www.tandfonline.com/doi/abs/10.1080/00029890.2002.11919857

[24] Csanky, L. (1976). Fast parallel matrix inversion algorithm. *SIAM J. Comput.* 5, 618–623.https://doi.org/10.1109/SFCS.1975.14

[25] López-Bonilla, J., Lucas-Bravo, A. & Vidal-Beltrán, S. (2019). Newton's formula and the inverse of a triangular matrix. *Studies in Nonlinear Sci.*, 4(2), 17–18.

[26] Wang, G., Wei, Y. & Qiao, S. (2018) *Generalized Inverses: Theory and Computations*, Springer, Singapore.https://link.springer.com/content/pdf/10.1007/978-981-13-0146-9.pdf

[27] Faddeev, D. K. & Sominsky, I. S. (1949). Collection of Problems on Higher Algebra. Moscow.

[28] Faddeeva, V.N. (1959). *Computational Methods of Linear Algebra*. Dover, New York.https://www.sidalc.net/search/Record/KOHA-OAI-TEST:9312/Description

[29] Faddeev, D.K. (1963). *Methods in Linear Algebra*. W. H. Freeman, San Francisco, USA.

[30] Gower, J.C. (1980). A modified Leverrier-Faddeev algorithm for matrices with multiple eigenvalues. *Linear Algebra and its Applications*, 31(1), 61–70.https://doi.org/10.1016/0024-3795(80)90206-2

[31] Helmberg, G., Wagner, P. & Veltkamp, G. (1993). On Faddeev-Leverrier's method for the computation of the characteristic polynomial of a matrix and of eigenvectors. *Linear Algebra and its Applications*, 185, 219–233.https://core.ac.uk/download/pdf/81192811.pdf

[32] Shui-Hung Hou (1998) On the Leverrier-Faddeev algorithm. *Electronic Proc. of Asia Tech. Conf. in Maths*.https://atcm.mathandtech.org/EP/1998/ATCMP002/paper.pdf

[33] Shui-Hung Hou (1998). A simple proof of the Leverrier-Faddeev characteristic polynomial algorithm, *SIAM Rev.*, 40(3), 706–709.https://doi.org/10.1137/S003614459732076X

[34] López-Bonilla, J., Morales, J., Ovando, G. & Ramírez, E. (2006). Leverrier-Faddeev's algorithm applied to spacetimes of class one. Proc. Pakistan Acad. Sci., 43(1), 47–50.https://paspk.org/wp-content/uploads/proceedings/43%20No.%201/a6fac9cf43-1-P47-50.pdf
[35] Caltenco, J. H., López-Bonilla, J. & Peña-Rivero, R. (2007). Characteristic polynomial of A and Faddeev's method for A-1, Educatia Matematica, 3(1-2), 107–112.http://www.cpgg.ufba.br/pessoal/reynam/Curso_HPC_2016_1/projetos/Faddeev.pdf
[36] López-Bonilla, J., Torres-Silva, H. & Vidal-Beltrán, S.(2018). On the Faddeev-Sominsky's algorithm. World Scientific News, 106, 238-244.https://agro.icm.edu.pl/agro/element/bwmeta1.element.agro-81a39821-7ea1-4591-9a41-2f539d8131ae
[37] http://mobiusfunction.wordpress.com/2010/08/07/the-inverse-of-a-triangular-matrix
[38] López-Bonilla, J. & Miranda-Sánchez, I. (2020). Inverse of a lower triangular matrix, Studies in Nonlinear Sci., 5(4), 57–58.https://idosi.org/sns/5(4)20/4.pdf
[39] http://mobiusfunction.wordpress.com/2010/12/08/the-inverse-of-triangular-matrix-as-a-binomial-series/
[40] Malenfant, J. (2011). Finite, Closed-Form Expressions for the Partition Function and for Euler, Bernoulli and Stirling Numbers. arXiv: 1103.1585v6 [math.NT] 24 May 2011. https://arxiv.org/abs/1103.1585
[41] López-Bonilla, J. & Vidal-Beltrán, S. (2021). On the Jha and Malenfant formulae for the partition function. Comput. Appl. Math. Sci., 6(1), 21–22.
[42] Bulnes, J. D., Kim, T., López-Bonilla, J. & Vidal-Beltrán, S. (2023). On the partition function. Proc. Jangjeon Math. Soc. 26(1), 1–9.https://www.researchgate.net
[43] Berkovich, A. & Uncu, A. K. (2019) Elementary polynomial identities involving q-trinomial coefficients. *Annals of Combinatorics*, 23, 549–560.https://doi.org/10.1007/s00026-019-00445-8
[44] Hirschhorn, M. D. (1999) Another short proof of Ramanujan's mod 5 partition congruences, and more. *Amer. Math. Monthly*, 106(6), 580–583.https://doi.org/10.1080/00029890.1999.12005087
[45] http://mathworld.wolfram.com/PartitionFunctionP.html
[46] Birmajer, J. B. Gil, Weiner, M. D. (2014) *Linear Recurrence sequences and their convolutions via Bell polynomials*. arXiv: 1405.7727v2 [math.CO] 26 Nov.https://doi.org/10.48550/arXiv.1405.7727
[47] Vein, R. & Dale, P. (1999). *Determinants and Their Applications in Mathematical Physics*, Springer-Verlag, New York.
[48] Gould, H. W. (2010). *Combinatorial Identities. Table I: Intermediate Techniques for Summing Finite Series*. Edited and compiled by J. Quaintance, May 3.