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A Variational Approach to the Hydrodynamics of Immiscible Viscous Flow in Vertical Pipes

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Abstract

In this study, a dedicated variational framework is proposed to model the steady-state transport of a non-mixing mass within a perfectly circular pipe. By applying consistent mathematical analysis, analytical solutions are obtained for key physical parameters of such systems, including the velocity distribution across the pipe radius for both vertical and horizontal orientations, the volumetric flow rate (derived in the form of the Hagen-Poiseuille relation), and the shear stress along the pipe wall. These results are particularly relevant to applications such as gas-lift oil well production, water-cut and sand-producing wells, and various two-phase or multiphase flow regimes. In deep hydrocarbon wells, the temperature typically increases by approximately 1 °C for every 10 m of depth; for wells extending several kilometers, this effect makes temperature a significant factor influencing flow behavior. As fluids rise from the wellbore to the surface, cooling occurs in a non-uniform manner, causing even a homogeneous fluid at the outlet to behave like a non-mixing system due to temperature-dependent variations in viscosity and density. Consequently, the flow can be treated as that of a non-mixing viscous medium. These considerations, along with associated operational implications, form the basis of the analysis presented in this article.

Keywords: Immiscible liquids, Bubbly flow, Incompressible fluids, Velocity profile, Viscous flow, Shear stress.

1 | Introduction

The study of multiphase flows in vertical conduits has long been a central theme in fluid mechanics and energy engineering due to its direct relevance to oil, gas, and chemical industries. In hydrocarbon production, crude oil is rarely extracted in a pure state; it is typically accompanied by water, natural gas, and sometimes solid impurities such as sand or clay. As these multiphase mixtures ascend through wellbores or pipelines, they exhibit diverse flow regimes, ranging from bubbly and slug flows to annular and mist flows, each characterized by unique transport dynamics and hydrodynamic interactions. Traditional modeling approaches,

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including the Navier–Stokes–Cahn–Hilliard framework, have provided valuable insights into immiscible two-phase flow; however, these approaches face challenges when extended to mixtures involving solid suspensions or non-uniformly heated liquids, which are increasingly encountered in modern oil and gas operations. Vertical pipe systems present a particularly advantageous setting for theoretical analysis due to their symmetry and frequent use in industrial applications such as gas-lift production, nuclear reactor cooling systems, and multiphase chemical reactors. Accurate modeling of such systems is crucial for predicting pressure drops, velocity distributions, and shear stresses, which directly influence energy efficiency, safety, and operational costs. In this context, variational methods offer a promising analytical framework for capturing the complex interactions of immiscible viscous components under steady-state conditions. This article develops a novel variational approach to describe immiscible viscous flows in vertical pipes. The proposed framework provides analytical expressions for velocity profiles, volumetric flow rates, and shear stresses, taking into account viscosity–density relationships and pressure-driven transport. By comparing theoretical results with experimental data from well-established studies, the research demonstrates both the applicability and accuracy of the approach, providing a solid foundation for future advancements in multiphase flow modeling.

This research presents a comprehensive variational framework for analyzing immiscible viscous flow in vertical pipes, with direct applications to gas-lift oil production and other multiphase transport processes. The work begins by formulating the governing equations based on the Navier–Stokes system, introducing a proportional relationship between the viscosity and density of the immiscible components. The analysis proceeds through a carefully constructed variational problem, leading to Euler–Lagrange equations whose solutions describe the velocity distribution and viscosity profile across the pipe radius. Key contributions include the derivation of analytical expressions for velocity fields, wall shear stress, and volumetric flow rates. A generalized Hagen–Poiseuille–type law is established for multiphase mixtures, marking a significant extension of classical laminar flow theory to complex, immiscible systems. The findings reveal that high-viscosity components concentrate near the pipe walls, where they move more slowly, while less viscous components dominate the central region of the flow. The study also addresses the practical conditions encountered in deep hydrocarbon wells, such as temperature gradients with depth, which result in non-uniform viscosity and density distributions. This feature allows the model to capture scenarios where fluids behave as non-mixing media due to thermal effects. Validation against experimental results reported by Kashinsky and Nakoryakov confirms the accuracy of the theoretical predictions, demonstrating close agreement with observed one-dimensional two-phase flow behavior.

The theoretical framework and analytical results presented in this article build upon a wide range of established works in fluid dynamics and applied mathematics. Foundational principles of incompressible fluid mechanics are grounded in Batchelor's classic text An Introduction to Fluid Dynamics [1], which provides the fundamental background for the governing equations used here. Historical developments of laminar pipe flow, particularly through Poiseuille's law, have been extensively documented by Sutera and Skalak [2], forming the basis for understanding viscous transport phenomena. More recent experimental studies, such as those by Arafin and Rahman [3], expand this foundation by presenting velocity profiles of crude oils and other liquids, which directly support the validation of the current variational model. Similarly, the investigations of Sotgia, Tartarini, and Stalio [4] into oil–water flow regimes and pressure drop reductions have provided important comparative insights for multiphase immiscible mixtures. In relation to bubbly flows, Kashinsky, Timkin, and Cartellier [5] and Nakoryakov et al. [6] offer benchmark experimental results in vertical conduits, which were directly used to verify the predictions of this research. In parallel, contemporary mathematical methods have been employed to broaden the theoretical background. For example, Juraev, Mammadzada, and Israr [7] have demonstrated applications of ill-posed problems in mathematical physics, providing analytical depth relevant to the variational approach adopted here. Complementary contributions by Shafiyeva, Ibrahimov, and Juraev [8] on the comparison of numerical methods for solving ODEs further strengthen the methodological framework. Finally, the work of Bulnes et al. [9] on Hamilton–Jacobi and Schrödinger formulations underscores the broader analytical versatility of variational techniques, thereby linking classical hydrodynamic models with more generalized mathematical formulations.

In the oil and gas industry, multiphase mixtures, typically consisting of oil, gas, water, and occasionally suspended solids or formation sand, are often transported through well tubing, exhibiting various flow regimes. The study of such multiphase flows has been the subject of extensive research due to their complexity and industrial significance [1], [2]. When gas and liquid phases flow simultaneously within a conduit, the distribution patterns of each phase vary according to fluid properties, flow rates, and the pipe's geometry or curvature. Commonly, five primary gas-liquid flow regimes are identified: bubbly flow, slug flow, semi-annular flow, annular flow, and mist flow.

In vertical gas–liquid transport, introducing gas at a low rate from the base of the column generates a dispersed field of small bubbles within the continuous liquid phase, characteristic of bubbly flow. This regime is prevalent in both static and moving liquids and is frequently encountered in practice due to its wide range of applications. Of particular interest is the analysis of bubbly flows in vertical standpipes operating under high gas load conditions. Two main factors drive this interest: 1) vertical conduits are common in a broad range of technical systems, including nuclear reactors, and 2) vertical pipe flow offers a relatively simple symmetry that facilitates analysis. The same mechanism is exploited in gas-lift oil production operations.

Crude oil extracted from wellheads often contains entrained sand, clay, or other granular particles, making its flow behavior more complex than that of crude oil without these particles. Similar conditions arise during the transport of hydrocarbons in long-distance pipelines [3]. From a theoretical perspective, the Navier-Stokes-Cahn-Hilliard framework serves as a diffuse interface model for immiscible two-phase flow of viscous, incompressible fluids. Extensive research exists on vertical pipe flows; however, the present work extends beyond traditional liquid-liquid or gas-liquid systems, such as oil-water [4] or oil-gas mixtures used in gas-lift methods, to also consider the transport of liquids containing suspended solids.

The primary objective of this study is to analyze steady multiphase flows, including liquid-liquid mixtures, liquid-solid suspensions, gas-liquid bubbly flows, and non-uniformly heated single-phase liquids under a specified pressure drop. A continuous media approach is developed to estimate key characteristics such as velocity profiles across the pipe cross-section, mean flow velocity, and wall shear stress. The model incorporates inlet, outlet, and wall boundaries, with inflow driven by high bottom-hole pressure, causing upward transport from deep zones to the wellhead.

Within this framework, new variational and theoretical foundations for describing the transport of complex mixtures in vertical circular pipes are presented. The derived analytical solutions represent one of the earliest attempts to predict velocity profiles, pressure drops, shear stresses, and volumetric flow rates, analogous to the Hagen-Poiseuille law, while demonstrating strong agreement with the experimental results for one-dimensional two-phase flows reported by Kashinsky et al. [5] and Nakoryakov et al. [6]. These results offer valuable theoretical support for future experimental studies investigating the hydrodynamics of gas-liquid flows.

2 | Field Equations and Formulation

In order to investigate the hydrodynamics of immiscible viscous flows in vertical pipes, it is necessary to establish a rigorous system of governing equations. The formulation begins with the Navier–Stokes equations for incompressible viscous fluids, which serve as the fundamental basis for describing momentum and continuity in multiphase transport. A key assumption adopted in this work is that the viscosity and density of the immiscible components are proportional, a simplification that allows the system to be reduced to a more analytically tractable variational problem. By introducing this proportionality, the model captures the layered distribution of high- and low-viscosity fluids, which significantly influences the velocity profiles across the pipe's cross-section. Boundary conditions are defined to reflect realistic wellbore conditions, with no-slip assumptions at the pipe wall and pressure-driven inflow at the base. The mathematical framework is then reformulated into a variational form, which enables the derivation of Euler–Lagrange equations for velocity and viscosity distribution. This approach not only ensures mathematical consistency but also aligns with experimental observations, where denser and more viscous fluids tend to accumulate near the pipe walls, while lighter components dominate the central region. The field equations further incorporate gravitational effects, as hydrostatic pressure gradients inherently influence vertical flow. The balance between upward pressure forces and downward gravitational forces is crucial in predicting the feasibility of flow in deep wells, particularly for heavy crude oils. Moreover, the developed formulation allows the calculation of essential transport parameters, including pressure drop, kinetic energy distribution, and shear stress along the pipe walls. The resulting equations thus provide a complete analytical framework that bridges theoretical hydrodynamics with practical applications in oil and gas engineering.

We consider the upward transport of a liquid mixture within a vertical pipe, such as flow from deep well zones to the surface through the wellbore. The motion is assumed to be one-dimensional and parallel to the pipe axis. The governing framework is based on the Navier-Stokes equations, with the assumption of a proportional relationship between the viscosity and density of the immiscible components. Specifically, when the constituents are arranged in ascending order of viscosity, their densities are assumed to increase proportionally,

$$\rho(r) = \alpha\mu(r), \quad 0 < r < R.$$

where α is the proportionality factor.

The movement and continuity equations for the process are expressed as follows:

$$\rho(r)\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) + \frac{\partial p}{\partial x} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu(r)\frac{\partial v}{\partial r}\right) + \rho(r)g,$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho(r)v) = 0, \quad 0 \le r \le R.$$

Where $v(x, y, z)$ is the magnitude of velocity at the point (x, y, z) at the moment t, the p(x) is the pressure, we assume it does not vary on the cross-section of the pipe. Since the steady flow is considered, we have to set $\frac{\partial v}{\partial t} = 0$, $\frac{\partial p}{\partial t} = 0$, it follows from the continuity equation $\frac{\partial v}{\partial x} = 0$.

Therefore, the movement equation for the process reads as

$$\frac{\partial p}{\partial x} = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu(r)\frac{\partial v}{\partial r}\right) + \rho(r)g, 0 \le r \le R,$$

Assuming the nonslip condition

$$v(R) = 0.$$

3 | The Velocity Profile, Pressure Drop and Energy Expression

From the governing equations established in the previous section, the expression for the pressure drop along the vertical pipe can be directly obtained. The velocity distribution across the pipe radius takes the general form:

$$p(x) = p(0) + \frac{p(H) - p(0)}{H}x, \quad 0 < x < H,$$

$$v(r) = \int_r^R \frac{1}{z\mu(z)}\left(\int_0^z \left(\frac{p(H) - p(0)}{H} - g\rho(t)\right) t dt\right) dz, \quad 0 < r < R.$$

Denoting

$$G(z) = \int_0^z \left(\frac{p(H) - p(0)}{H} - \rho(t)g\right) t dt, 0 < z < R,$$

for the kinetic energy, we have the expression

$$E = \pi L \int_0^R \left(\int_\theta^R \frac{G(z)}{z\mu(z)}\right)^2 \rho(\theta)\theta d\theta.$$

To solve the particle distribution, we have settled the following auxiliary variational problem.

$$\int_0^R \left[M\left(-\frac{t^2y^2}{y'}\right) - \left(\frac{t^2}{y'} + u'\right)\lambda(t)\right] dt \rightarrow \min.$$

The functions $y(t), u(t), \lambda(t)$ are such that

$$-\int_0^R \frac{t^2}{y'} dt = \frac{\overline{\rho}R^2}{\alpha},$$

$$M = \frac{\pi L\alpha}{4}(\kappa - \overline{\rho}\, g)^2,$$

$$y(t) = \int_t^R \frac{z\, dz}{\mu(z)}.$$

The Euler-Lagrange equation for the given functional is the following system of equations.

$$-\frac{2t^2y(t)}{y'} - \frac{d}{dt}\left(\frac{t^2y^2(t)}{(y'(t))^2} + \frac{1}{M}\frac{t^2\lambda(t)}{(y'(t))^2}\right) = 0.$$

Further solving this equation yields the expressions

$$y(t) = k \cdot \tanh\left(C_2\left(1 - \frac{r}{R}\right)\right), \qquad 0 < r < R, \tag{1}$$

$$\mu(t) = \frac{tR}{kC_2}\cosh^2 C_2\left(1 - \frac{t}{R}\right), \qquad 0 < r < R, \tag{2}$$

for the velocity distribution over the diameter of the pipe. The viscosity profile over the radius is determined. The velocity profiles over the cross-section of the vertical pipe in the upward flow of the gas-liquid mixture are compared with the experimental results of Kashinsky et al. [5] and Nakoryakov et al. [6].

4 | Conlusion

This study presents analytical expressions for the velocity profile across a pipe cross-section, the wall shear stress, and the volumetric flow rate for upward transport in vertical circular pipes carrying various mixtures, including liquid-liquid systems, liquid-gas bubbly flows, liquid-solid suspensions, and non-uniformly heated single-phase liquids. The results show that high-viscosity components of the mixture tend to concentrate near the pipe wall, where they move more slowly than the less viscous core fluid.

A generalized Hagen-Poiseuille-type formula for the flow rate of such mixtures has been derived for the first time. The proposed theoretical framework also provides new empirical expressions for velocity distribution, which are consistent with experimental observations. Additionally, it is shown that the product of the average mixture density and gravitational acceleration can exceed the pressure gradient:

$$k - \rho g \cong \kappa - \bar{\rho} g,$$

$$k = (p(H) - p(0))/H.$$

This relation can be used to estimate the feasibility of extracting heavy oil under a given drawdown pressure. The methodology is original and can also be applied to systems where viscosity does not necessarily increase with density, as well as to mass transport problems involving non-liquid complex media.

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Author Contribution

Conceptualization, N.M.; Methodology, N.M.; Software, N.M.; Validation, N.M.; formal analysis, N.M.; investigation, N.M.; resources, N.M.; data maintenance, N.M.; writing-creating the initial design, N.M.; writing-reviewing and editing.

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Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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