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Integral Inequalities for Twice Differentiable Functions by Novel Fractional Operator

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Abstract

This paper presents new integral inequalities for twice differentiable (Λ, δ)-convex functions involving extended weighted fractional integrals. By leveraging a fundamental identity and employing Hölder, Power mean and Young inequalities under various convexity conditions on the derivative, we generalize and refine Hermite-Hadamard type inequalities. Applications to Riemann-Liouville, Caputo-Fabrizio and Atangana-Baleanu fractional integral operators demonstrate the broad applicability of the results.

Keywords: Hermite-Hadamard inequality, Convex functions, Integral inequalities, Extended weighted fractional integral.

1|Introduction

The Hermite-Hadamard inequality has been a cornerstone in the study of convex functions, providing accurate estimates for the average value of a function. In recent decades, the search for generalizations that encompass nonlocal operators and nonstandard geometries has driven the development of fractional calculus and new notions of convexity (see [1, 16, 17, 19]).

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The notion of convexity (Λ, δ) presented in this work is not only a mathematical abstraction, but a unified framework that allows us to recover classic results from Godunova-Levin, Dragomir, and others, under a single formalism. This simplifies the study of inequalities by allowing us to treat various families of functions simultaneously.

Definition 1. *Let D be a real interval and $\Lambda : [0,1] \times D^2 \to D$ be a function. We say that Λ is a weighted path on D if there exist real functions Φ and Ψ of class C^1 such that:*

- $[0,1] \subseteq \mathrm{Dom}(\Phi)$ *and* $[0,1] \subseteq \mathrm{Dom}(\Psi)$.
- $\Phi([0,1]) \subseteq [0,1]$ *and* $\Psi([0,1]) \subseteq [0,1]$.
- Φ *is strictly increasing on* $[0,1]$ *and* Ψ *is strictly decreasing on* $[0,1]$.
- $\Lambda(\varsigma, \vartheta, \rho) = \Phi(\varsigma)\vartheta + \Psi(\varsigma)\rho$ *with* $\frac{\partial \Lambda}{\partial \varsigma} \neq 0$.

For fixed values of ρ and ϑ, we denote $\Lambda_{\vartheta,\rho}$.

Definition 2. *Let $D \subseteq [0, +\infty)$ be a real interval, $\varphi : D \to [0, +\infty)$ a function, $\Lambda(\varsigma, \rho, \vartheta) = \Phi(\varsigma)\vartheta + \Psi(\varsigma)\rho$ is a weighted path on D and $\delta : [0,1] \to \mathrm{Dom}(\Phi) \cap \mathrm{Dom}(\Psi)$ a non-negative function with $\delta \neq 0$. If inequality*

$$\varphi\left(\Phi(\varsigma)\rho + m\Psi(\varsigma)\vartheta\right) \leq \Phi(\delta(\varsigma))\varphi(\rho) + m\Psi(\delta(\varsigma))\varphi(\vartheta)$$

is fulfilled for all $\rho, \vartheta \in D$ and $\varsigma \in [0,1]$ where $m \in [0,1]$. Then the function φ is called (Λ, δ)-convex of the first type on D.

Remark 1. *Considering $\Phi(\varsigma) = \varsigma$, $\Psi(\varsigma) = 1 - \varsigma$ and $\delta(\varsigma) = h^s(\varsigma)$ with $s \in (0,1]$, one obtains the modified (h, m)-convexity of the first type on D (see* [9], [10]*). Moreover, several other types of convexity discussed in* [20] *can be recovered as particular cases of Definition 2:*

- *h-convexity (for $\Phi(\varsigma) = \varsigma$ and $\Psi(\varsigma) = 1 - \varsigma$);*
- *m-convexity (by taking $\Phi(\varsigma) = \delta(\varsigma) = \varsigma$ and $\Psi(\varsigma) = 1 - \varsigma$);*
- *(α, m)-convexity (by establishing $\Phi(\varsigma) = \varsigma$, $\Psi(\varsigma) = 1 - \varsigma$ and $\delta(\varsigma) = \varsigma^\alpha$ with $\alpha \in [0,1]$);*
- *Partially h-convexity (by selecting $\Phi(\varsigma) = \varsigma$, $\Psi(\varsigma) = 1 - \varsigma$ and $m = 1$);*
- *Classical convexity (if $\Phi(\varsigma) = \delta(\varsigma) = \varsigma$, $\Psi(\varsigma) = 1 - \varsigma$ and $m = 1$).*

Definition 3. *Let $D \subseteq [0, +\infty)$ be a real interval, $\varphi : D \to [0, +\infty)$ a function, $\Lambda(\varsigma, \vartheta, \rho) = \Phi(\varsigma)\vartheta + \Psi(\varsigma)\rho$ is a weighted path on D and $\delta : [0,1] \to \mathrm{Dom}(\Phi) \cap \mathrm{Dom}(\Psi)$ a nonnegative function with $\delta \neq 0$. If inequality*

$$\varphi\left(\Phi(\varsigma)\vartheta + m\Psi(\varsigma)\rho\right) \leq \Phi(\delta^s(\varsigma))\varphi(\vartheta) + m\Psi^s(\delta(\varsigma))\varphi(\rho)$$

is fulfilled for all $\rho, \vartheta \in D$ and $\varsigma \in [0,1]$ where $m \in [0,1]$ and $s \in [-1,1]$. Then the function φ is called (Λ, δ)-convex of the second type on D.

Remark 2. *Substituting $\Phi(\varsigma) = \varsigma$ and $\Psi(\varsigma) = 1 - \varsigma$, the (Λ, δ)-convexity of the second type reduces to the modified (h, m)-convexity of the second type on D (see* [9], [10]*). In addition, several other types of convexity presented in* [20] *can be obtained as particular cases of Definition 4:*

- *s-convexity in the second sense (by considering $\Phi(\varsigma) = \delta(\varsigma) = \varsigma$, $\Psi(\varsigma) = 1 - \varsigma$ and $s \in (0,1]$);*
- *Godunova–Levin convexity (by setting $\Phi(\varsigma) = \delta(\varsigma) = \varsigma$, $\Psi(\varsigma) = 1 - \varsigma$, $s = -1$ and $m = 1$);*
- *P-convexity (for $\Phi(\varsigma) = \delta(\varsigma) = \varsigma$, $\Psi(\varsigma) = 1 - \varsigma$, $s = 0$ and $m = 1$).*

By putting $s = 1$, the convexities mentioned in Remark 3 can be derived by choosing appropriate Φ, Ψ and δ.

The power of the (Λ, δ)-convexity framework lies in its ability to generalize beyond simple linear combinations. The conceptual key is the double nonlinearity:

(1) The argument of φ is evaluated at a nonlinear combination: $\Phi(\varsigma)\vartheta + \Psi(\varsigma)\rho$

(2) The weights on the right-hand side of Definition 2 or Definition 4 are themselves transformed by the function δ (or its powers). Specifically, these weights become $\Phi(\delta(\varsigma))$ and $\Psi(\delta(\varsigma))$ in the first kind, and $\Phi(\delta^s(\varsigma))$ and $(\Psi(\delta(\varsigma)))^s$ in the second kind.

This allows modeling convex behaviors with respect to unconventional interpolation paths, such as:

- **Exponential path:**

  $\Phi(\varsigma) = \frac{e^{\varsigma}-1}{e-1}$ and $\Psi(\varsigma) = 1 - \frac{e^{\varsigma}-1}{e-1}$

  $\Phi(\varsigma) = 2^{\varsigma} - 1$ and $\Psi(\varsigma) = 2 - 2^{\varsigma}$

- **Trigonometric path:**

  $\Phi(\varsigma) = \frac{\sin\left(\frac{\pi}{3}\varsigma\right)}{\sin\left(\frac{\pi}{3}\right)}$ and $\Psi(\varsigma) = 1 - \frac{\sin\left(\frac{\pi}{3}\varsigma\right)}{\sin\left(\frac{\pi}{3}\right)}$

  $\Phi(\varsigma) = \frac{\cos\left(\frac{\pi}{6}\varsigma+\frac{3\pi}{2}\right)}{\cos\left(\frac{5\pi}{3}\right)}$ and $\Psi(\varsigma) = 1 - \frac{\cos\left(\frac{\pi}{6}\varsigma+\frac{3\pi}{2}\right)}{\cos\left(\frac{5\pi}{3}\right)}$

  $\Phi(\varsigma) = tan(\frac{\pi}{4}\varsigma)$ and $\Psi(\varsigma) = 1 - tan(\frac{\pi}{4}\varsigma)$

  $\Phi(\varsigma) = \frac{1-\cos(\pi\varsigma)}{2}$ and $\Psi(\varsigma) = \frac{1}{2} + \frac{\cos(\pi\varsigma)}{2}$

- **Logarithmic path:** $\Phi(\varsigma) = \frac{\ln(1+\varsigma)}{\ln 2}$ and $\Psi(\varsigma) = 1 - \frac{\ln(1+\varsigma)}{\ln 2}$

- **Algebraic path:** $\Phi(\varsigma) = \frac{\lambda\varsigma^r}{\lambda+1} + \frac{\varsigma}{\lambda+1}$ and $\Psi(\varsigma) = 1 - \Phi(\varsigma)$ with $r \geq 1$ and $\lambda \geq 0$

- **Rational path:** $\Phi(\varsigma) = \frac{2\varsigma}{1+\varsigma}$ and $\Psi(\varsigma) = 1 - \frac{2\varsigma}{1+\varsigma}$

It is thus possible to engineer bespoke forms of convexity, fine-tuned to the intricacies of particular problems.

Unlike traditional approaches that use fixed kernels, our use of extended heavy fractional integrals allows for unprecedented flexibility, accommodating operators with singular and non-singular kernels, such as Riemann-Liouville, Caputo-Fabrizio and Atangana-Baleanu.

Definition 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $g : \mathbb{R} \to [0, \infty)$ be an integrable function. The convolution of f and g is defined by*

$$(f * g)(\varsigma) = \int_{-\infty}^{+\infty} f(\varsigma - z)g(z)\,dz.$$

Definition 5. *(see [22]) For $\varphi \in L[\alpha, \beta]$ and $t \in (\alpha, \beta)$, let $w : D \to \mathbb{R}$ be a positive continuous function whose second derivative is integrable on the interior of D. The right and left-sided weighted fractional integral operators are defined respectively by*

$$J^w_{\alpha^+}\varphi(t) = \int_\alpha^t w''\left(\frac{t-\varsigma}{\beta-\alpha}\right)\varphi(\varsigma)\,d\varsigma,$$
$$J^w_{\beta^-}\varphi(t) = \int_t^\beta w''\left(\frac{\varsigma-t}{\beta-\alpha}\right)\varphi(\varsigma)\,d\varsigma.$$

Definition 6. *(see [27]) Let $\varphi \in L[\alpha, \beta]$. The Riemann–Liouville fractional integrals $I^\eta_{\alpha+}\varphi$ and $I^\eta_{\beta-}\varphi$ of order $\eta > 0$ with $\alpha \geq 0$ are defined by*

$$I^\eta_{\alpha+}\varphi(t) = \frac{1}{\Gamma(\eta)}\int_\alpha^t (t-\varsigma)^{\eta-1}\varphi(\varsigma)\,d\varsigma, \qquad t > \alpha,$$

and

$$I^\eta_{\beta-}\varphi(t) = \frac{1}{\Gamma(\eta)}\int_t^\beta (\varsigma-t)^{\eta-1}\varphi(\varsigma)\,d\varsigma, \qquad t < \beta,$$

respectively. Here, $\Gamma(\eta)$ is the Gamma function and $I^0_{\alpha+}\varphi(t) = I^0_{\beta-}\varphi(t) = \varphi(t)$.

Definition 7. *(see [14]) Let $\varphi \in H^1(\alpha,\beta)$, $\alpha<\beta$ and $\eta \in [0,1]$, then we define the Caputo-Fabrizio's fractional integral (left and right respectively):*

$$\left({}_{\alpha}^{CF}I^{\eta}\varphi\right)(t)=\frac{1-\eta}{N(\eta)}\varphi(t)+\frac{\eta}{N(\eta)}\int_{\alpha}^{t}\varphi(\varsigma)\,d\varsigma,$$

$$\left({}^{CF}I_{\beta}^{\eta}\varphi\right)(t)=\frac{1-\eta}{N(\eta)}\varphi(t)+\frac{\eta}{N(\eta)}\int_{t}^{\beta}\varphi(\varsigma)\,d\varsigma,$$

with $H^1(\alpha,\beta)=\{\,\varphi\in L^2(\alpha,\beta)\ :\ \varphi'\in L^2(\alpha,\beta)\,\}$ and $N(\eta)$ a normalization function satisfying $N(0)=N(1)=1$.

The Atangana-Baleanu (AB) integral operators are formulated as follows:

Definition 8. *(see [8]) For $\beta>\alpha$ and $\eta\in[0,1]$, the left-sided fractional integral associated with the novel fractional derivative featuring a non-local kernel, for a function $\varphi\in H^1(\alpha,\beta)$, is defined as:*

$${}^{AB}I_{\alpha+}^{\eta}\varphi(t)=\frac{1-\eta}{N(\eta)}\varphi(t)+\frac{\eta}{N(\eta)\Gamma(\eta)}\int_{\alpha}^{t}\frac{\varphi(\mu)}{(\mu-\alpha)^{1-\eta}}\,d\mu.$$

Definition 9. *(see [7]) The right-sided fractional integral corresponding to the fractional derivative with a non-local kernel, for a function $\varphi\in H^1(\alpha,\beta)$, is given by:*

$${}^{AB}I_{\beta-}^{\eta}\varphi(t)=\frac{1-\eta}{N(\eta)}\varphi(t)+\frac{\eta}{N(\eta)\Gamma(\eta)}\int_{t}^{\beta}\frac{\varphi(\mu)}{(\beta-\mu)^{1-\eta}}\,d\mu,$$

in which $\beta>\alpha$ and $\eta\in[0,1]$.

Next we define the extended weighted fractional integral that we will use in our work.

Definition 10. *Let $\varphi\in L[\alpha,\beta]$, $w:D\to\mathbb{R}$ be a continuous and positive function with its second derivative integrable on D°, $\Lambda(\varsigma,\vartheta,\rho)$ is a weighted path on D and $\rho,\vartheta\in[\alpha,\beta]$. Then the extended weighted fractional integral is defined as:*

$$\mathcal{L}_{\varphi}^{w}\Lambda_{\vartheta,\rho}(\varsigma)=\int_{0}^{1}w''(\varsigma)\varphi\left(\Lambda_{\vartheta,\rho}(\varsigma)\right)\,d\varsigma.$$

Remark 3. *The inclusion of second derivative of the weight function w arises from the inherent nature of the problem. Alternatively, the first derivative may also be considered.*

Remark 4. *Consider the linear interpolation $\Lambda_{\vartheta,\rho}(\varsigma)=(1-\varsigma)\rho+\varsigma\vartheta$ with $\rho<\vartheta$. By making the substitution $x=\rho+(\vartheta-\rho)\varsigma$, the extended weighted fractional integral of φ coincides with $\frac{1}{\vartheta-\rho}\,J_{\rho+}^{w}g(\vartheta)$.*

Similarly, for the linear interpolation in the reverse order $\Lambda_{\rho,\vartheta}(\varsigma)=(1-\varsigma)\vartheta+\varsigma\rho$ where $\rho<\vartheta$ and the substitution $y=\vartheta+(\rho-\vartheta)\varsigma$, the extended weighted fractional integral of φ coincides with $\frac{1}{\vartheta-\rho}\,J_{\vartheta-}^{w}\varphi(\rho)$.

This article establishes novel integral inequalities within the framework of second-type (Λ,δ)-convex functions by employing extended weighted fractional integrals. Various implications for fractional integrals of the Riemann-Liouville and Atangana-Baleanu types are explored throughout the study.

2|First Result

We now turn to the analysis of the Hermite–Hadamard inequality within the framework of extended weighted fractional integrals. Our goal is to derive new bounds under the assumption of (Λ,δ)-convexity of the second type, which generalizes several classical convexity notions discussed in the previous section.

In what follows, we establish a Hermite-Hadamard type inequality adapted to the extended weighted integral operator introduced above.

Theorem 1. *Let $\alpha, \beta \in D^o$ with $\alpha < \beta$ and let $\varphi : D \to \mathbb{R}$ be a function that satisfies the (Λ, δ)- convexity condition of the second type. Consider a weight function $w : [0,1] \to \mathbb{R}$ and $\Lambda(\varsigma, \beta, \alpha) = \Phi(\varsigma)\beta + \Psi(\varsigma)\alpha$ a weighted path on D such that $\Phi(\frac{1}{2}) = \Psi(\frac{1}{2}) = \frac{1}{2}$. Suppose that $w''(\varsigma) \geq 0$ for all $\varsigma \in (0,1)$, $\left[\alpha, \frac{\beta}{m}\right] \subset D^o$, $m \in [0,1]$ and $s \in [-1,1]$. Under these assumptions, we deduce:*

$$\begin{aligned}\varphi\left(\frac{\alpha+\beta}{2}\right)\left(w'(1) - w'(0)\right) &\leq \Phi\left(\delta^s\left(\tfrac{1}{2}\right)\right)\mathcal{L}_{\varphi}^{w}\bar{\Lambda}_{\beta,\alpha}(\varsigma) + \left(\Psi\left(\delta\left(\tfrac{1}{2}\right)\right)\right)^s \mathcal{L}_{\varphi}^{w}\bar{\Lambda}_{\alpha,\beta}(\varsigma) \\ &\leq \left[\Phi\left(\delta^s\left(\tfrac{1}{2}\right)\right)\varphi(\beta) + \left(\Psi\left(\delta\left(\tfrac{1}{2}\right)\right)\right)^s \varphi(\alpha)\right]\int_0^1 w''(\varsigma)\,\Phi(\delta^s(\varsigma))\,d\varsigma \\ &\quad + m\left[\Phi\left(\delta^s\left(\tfrac{1}{2}\right)\right)\varphi\left(\tfrac{\alpha}{m}\right) + \left(\Psi\left(\delta\left(\tfrac{1}{2}\right)\right)\right)^s \varphi\left(\frac{\beta}{m}\right)\right]\int_0^1 w''(\varsigma)\left(1 - \Phi(\delta(\varsigma))\right)^s d\varsigma,\end{aligned} \tag{1}$$

where $\bar{\Lambda}_{\beta,\alpha}(\varsigma) = \Phi(\varsigma)\beta + (1 - \Phi(\varsigma))\,\alpha$.

Proof: For $\alpha < \beta$, $\varsigma = \frac{1}{2}$ and $m = 1$ in Definition 4, we get:

$$\varphi\left(\frac{\alpha+\beta}{2}\right) = \varphi\left(\Phi\left(\frac{1}{2}\right)\alpha + \Psi\left(\frac{1}{2}\right)\beta\right) \leq \Phi\left(\delta^s\left(\tfrac{1}{2}\right)\right)\varphi(\alpha) + \left(\Psi\left(\delta\left(\tfrac{1}{2}\right)\right)\right)^s \varphi(\beta).$$

Taking into account $\bar{\Lambda}_{\alpha,\beta}(\varsigma) = \Phi(\varsigma)\alpha + (1 - \Phi(\varsigma))\,\beta$, we find:

$$\varphi\left(\frac{\alpha+\beta}{2}\right) = \varphi\left(\frac{\bar{\Lambda}_{\beta,\alpha}(\varsigma) + \bar{\Lambda}_{\alpha,\beta}(\varsigma)}{2}\right) \leq \Phi\left(\delta^s\left(\tfrac{1}{2}\right)\right)\varphi\left(\bar{\Lambda}_{\beta,\alpha}(\varsigma)\right) + \left(\Psi\left(\delta\left(\tfrac{1}{2}\right)\right)\right)^s \varphi\left(\bar{\Lambda}_{\alpha,\beta}(\varsigma)\right).$$

Multiplying both members of the previous inequality by $w''(\varsigma)$, integrating over the interval $[0,1]$ with respect to ς, we obtain:

$$\begin{aligned}\varphi\left(\frac{\alpha+\beta}{2}\right)\int_0^1 w''(\varsigma)\,d\varsigma &\leq \Phi\left(\delta^s\left(\tfrac{1}{2}\right)\right)\int_0^1 w''(\varsigma)\varphi\left(\bar{\Lambda}_{\beta,\alpha}(\varsigma)\right)d\varsigma + \Psi\left(\delta\left(\tfrac{1}{2}\right)\right)^s \int_0^1 w''(\varsigma)\varphi\left(\bar{\Lambda}_{\alpha,\beta}(\varsigma)\right)d\varsigma \\ &= \Phi\left(\delta^s\left(\tfrac{1}{2}\right)\right)\mathcal{L}_{\varphi}^{w}\bar{\Lambda}_{\beta,\alpha}(\varsigma) + \Psi\left(\delta\left(\tfrac{1}{2}\right)\right)^s \mathcal{L}_{\varphi}^{w}\bar{\Lambda}_{\alpha,\beta}(\varsigma).\end{aligned}$$

In this way we have the first part of the desired inequality.

By means of second type (Λ, δ)- convexity property of φ, one has:

$$\varphi\left(\bar{\Lambda}_{\beta,\alpha}(\varsigma)\right) = \varphi\left(\Phi(\varsigma)\beta + m\left(1 - \Phi(\varsigma)\right)\tfrac{\alpha}{m}\right) \leq \Phi\left(\delta^s(\varsigma)\right)\varphi(\beta) + m\left(1 - \Phi\left(\delta(\varsigma)\right)\right)^s \varphi\left(\tfrac{\alpha}{m}\right). \tag{2}$$

In an analogous way,

$$\varphi\left(\bar{\Lambda}_{\alpha,\beta}(\varsigma)\right) \leq \Phi\left(\delta^s(\varsigma)\right)\varphi(\alpha) + m\left(1 - \Phi\left(\delta(\varsigma)\right)\right)^s \varphi\left(\frac{\beta}{m}\right). \tag{3}$$

Multiplying (2) by $\Phi\left(\delta^s\left(\frac{1}{2}\right)\right)$ and (3) by $\left(\Psi\left(\delta\left(\frac{1}{2}\right)\right)\right)^s$ and adding both results leads us to:

$$\begin{aligned}&\Phi\left(\delta^s\left(\tfrac{1}{2}\right)\right)\varphi\left(\bar{\Lambda}_{\beta,\alpha}(\varsigma)\right) + \left(\Psi\left(\delta\left(\tfrac{1}{2}\right)\right)\right)^s \varphi\left(\bar{\Lambda}_{\alpha,\beta}(\varsigma)\right) \\ &\leq \Phi\left(\delta^s\left(\tfrac{1}{2}\right)\right)\left[\Phi\left(\delta^s(\varsigma)\right)\varphi(\beta) + m\left(1 - \Phi\left(\delta(\varsigma)\right)\right)^s \varphi\left(\tfrac{\alpha}{m}\right)\right] \\ &\quad + \left(\Psi\left(\delta\left(\tfrac{1}{2}\right)\right)\right)^s \left[\Phi\left(\delta^s(\varsigma)\right)\varphi(\alpha) + m\left(1 - \Phi\left(\delta(\varsigma)\right)\right)^s \varphi\left(\frac{\beta}{m}\right)\right].\end{aligned}$$

Finally, multiplying by $w''(\varsigma)$, integrating over the interval $[0,1]$ with respect to ς and suitably rearranging the terms, we reach the right-hand side of (1). This concludes the proof. □

To illustrate the above theorem, we will perform a numerical verification. This theorem establishes an upper bound for the heavy integral operator $\mathcal{L}_{\varphi}^{w}$ based on the convexity of φ.

Example 1. *We select the following parameters.*

- $\varphi(\mu) = \mu^2$;
- $[\alpha, \beta] = [1, 2]$;
- $w(\nu) = \nu^2 + 1$, *for which* $w''(\nu) = 2$;
- $\Phi(\nu) = \nu$, $\Psi(\nu) = 1 - \nu$;
- $\delta(\nu) = \nu$, $m = s = 1$.

To begin with,

$$w'(1) - w'(0) = 2.$$

The corresponding weighted paths are given by

$$\bar{\Lambda}_{\beta,\alpha}(\nu) = 2\nu + (1 - \nu) = 1 + \nu,$$

and

$$\bar{\Lambda}_{\alpha,\beta}(\nu) = \nu + 2(1 - \nu) = 2 - \nu.$$

Consequently,

$$\mathcal{L}_{\varphi}^{w} \bar{\Lambda}_{\beta,\alpha}(\nu) = \int_0^1 2(1 + \nu)^2 \, d\nu = \frac{14}{3},$$

whereas

$$\mathcal{L}_{\varphi}^{w} \bar{\Lambda}_{\alpha,\beta}(\nu) = \int_0^1 2(2 - \nu)^2 \, d\nu = \frac{14}{3}.$$

Since

$$\delta\left(\frac{1}{2}\right) = \frac{1}{2},$$

the middle expression in (1) *is evaluated as*

$$\frac{1}{2}\mathcal{L}_{\varphi}^{w} \bar{\Lambda}_{\beta,\alpha}(\nu) + \frac{1}{2}\mathcal{L}_{\varphi}^{w} \bar{\Lambda}_{\alpha,\beta}(\nu) = \frac{14}{3}.$$

On the other hand,

$$\varphi\left(\frac{\alpha + \beta}{2}\right) \big(w'(1) - w'(0)\big) = \varphi\left(\frac{3}{2}\right) \cdot 2 = \frac{9}{2}.$$

A simple calculation yields

$$\int_0^1 w''(\nu) \, \Phi(\delta^s(\nu)) \; d\nu = \int_0^1 2\nu \, d\nu = 1,$$

and

$$\int_0^1 w''(\nu) \, (1 - \Phi(\delta(\nu)))^s \; d\nu = \int_0^1 2(1 - \nu) \, d\nu = 1.$$

Accordingly, the right-hand side of (1) *is*

$$\left[\frac{1}{2}\varphi(2) + \frac{1}{2}\varphi(1)\right] + \left[\frac{1}{2}\varphi(1) + \frac{1}{2}\varphi(2)\right] = 5.$$

Therefore,

$$\boxed{\frac{9}{2} \le \frac{14}{3} \le 5}.$$

Corollary 1. *Let the hypotheses of Theorem 15 be satisfied for $\Phi(\varsigma) = \varsigma$ and $\Psi(\varsigma) = 1 - \varsigma$. Under this specification, one has, for all $\lambda \in \mathbb{N} \cup \{0\}$:*

$$\begin{aligned}
&\varphi\left(\frac{\alpha+\beta}{2}\right)(w'(1)-w'(0)) \leq \frac{\lambda+1}{\beta-\alpha}\left[\delta^s\left(\frac{1}{2}\right) J^w_{\alpha^+}\varphi\left(\tfrac{\beta+\lambda\alpha}{\lambda+1}\right) + \left(1-\delta\left(\frac{1}{2}\right)\right)^s J^w_{\beta-}\varphi\left(\tfrac{\alpha+\lambda\beta}{\lambda+1}\right)\right] \\
&\leq \left[\delta^s\left(\frac{1}{2}\right)\varphi(\beta) + \left(1-\delta\left(\frac{1}{2}\right)\right)^s\varphi(\alpha)\right]\int_0^1 w''(\varsigma)\,\delta^s\left(\frac{\varsigma}{\lambda+1}\right)d\varsigma \\
&\quad + m\left[\delta^s\left(\frac{1}{2}\right)\varphi\left(\frac{\alpha}{m}\right) + \left(1-\delta\left(\frac{1}{2}\right)\right)^s\varphi\left(\frac{\beta}{m}\right)\right]\int_0^1 w''(\varsigma)\left(1-\delta\left(\frac{\varsigma}{\lambda+1}\right)\right)^s d\varsigma.
\end{aligned} \tag{4}$$

Proof: Note that

$$\begin{aligned}
\varphi\left(\frac{\alpha+\beta}{2}\right) = \varphi\left(\frac{u+v}{2}\right) &\leq \delta^s\left(\frac{1}{2}\right)\varphi\left(\frac{\varsigma}{\lambda+1}\beta + \frac{\lambda+1-\varsigma}{\lambda+1}\alpha\right) \\
&\quad + \left(1-\delta\left(\frac{1}{2}\right)\right)^s\varphi\left(\frac{\varsigma}{\lambda+1}\alpha + \frac{\lambda+1-\varsigma}{\lambda+1}\beta\right),
\end{aligned}$$

where

$$u = \frac{\varsigma}{\lambda+1}\beta + \frac{\lambda+1-\varsigma}{\lambda+1}\alpha, \qquad v = \frac{\varsigma}{\lambda+1}\alpha + \frac{\lambda+1-\varsigma}{\lambda+1}\beta,$$

with $\varsigma \in [0,1]$.

Integrating both sides against $w''(\varsigma)$ over $[0,1]$ gives:

$$\begin{aligned}
\varphi\left(\frac{\alpha+\beta}{2}\right)\int_0^1 w''(\varsigma)\,d\varsigma &\leq \delta^s\left(\frac{1}{2}\right)\int_0^1 w''(\varsigma)\varphi\left(\frac{\varsigma}{\lambda+1}\beta + \frac{\lambda+1-\varsigma}{\lambda+1}\alpha\right)d\varsigma \\
&\quad + \left(1-\delta\left(\frac{1}{2}\right)\right)^s\int_0^1 w''(\varsigma)\varphi\left(\frac{\varsigma}{\lambda+1}\alpha + \frac{\lambda+1-\varsigma}{\lambda+1}\beta\right)d\varsigma.
\end{aligned}$$

After performing the change of variables induced by u and v, we obtain:

$$\begin{aligned}
&\varphi\left(\frac{\alpha+\beta}{2}\right)(w'(1)-w'(0)) \\
&\leq \frac{\lambda+1}{\beta-\alpha}\left[h^s\left(\frac{1}{2}\right)\int_\alpha^{\frac{\beta+\lambda\alpha}{\lambda+1}} w''\left[\frac{(u-\alpha)}{\frac{\beta-\alpha}{\lambda+1}}\right]\varphi(u)\,du\right. \\
&\quad \left. + \left(1-h\left(\frac{1}{2}\right)\right)^s\int_{\frac{\alpha+\lambda\beta}{\lambda+1}}^{\beta} w''\left[\frac{(\beta-v)}{\frac{\beta-\alpha}{\lambda+1}}\right]\varphi(v)\,dv\right] \\
&= \frac{\lambda+1}{\beta-\alpha}\left[\delta^s\left(\frac{1}{2}\right)J^w_{\alpha^+}\varphi\left(\tfrac{\beta+\lambda\alpha}{\lambda+1}\right) + \left(1-\delta\left(\frac{1}{2}\right)\right)^s J^w_{\beta-}\varphi\left(\tfrac{\alpha+\lambda\beta}{\lambda+1}\right)\right].
\end{aligned}$$

This verifies the first bound in (4). To proceed further, we invoke the (h,m)-convexity of φ:

$$\begin{aligned}
&\varphi\left(\frac{\varsigma}{\lambda+1}\beta + \frac{\lambda+1-\varsigma}{\lambda+1}\alpha\right) \\
&\leq \delta^s\left(\frac{\varsigma}{\lambda+1}\right)\varphi(\beta) + m\left(1-\delta\left(\frac{\varsigma}{\lambda+1}\right)\right)^s\varphi\left(\frac{\alpha}{m}\right).
\end{aligned} \tag{5}$$

Analogously,

$$\begin{aligned}
&\varphi\left(\frac{\varsigma}{\lambda+1}\alpha+\frac{\lambda+1-\varsigma}{\lambda+1}\beta\right) \\
&\leq \delta^s\left(\frac{\varsigma}{\lambda+1}\right)\varphi(\alpha)+m\left(1-\delta\left(\frac{\varsigma}{\lambda+1}\right)\right)^s\varphi\left(\frac{\beta}{m}\right).
\end{aligned}\tag{6}$$

Multiplying (5) by $\delta^s\left(\frac{1}{2}\right)$ and (6) by $\left(1-\delta\left(\frac{1}{2}\right)\right)^s$ and adding both results, we find:

$$\begin{aligned}
&\delta^s\left(\frac{1}{2}\right)\varphi\left(\frac{\varsigma}{\lambda+1}\beta+\frac{\lambda+1-\varsigma}{\lambda+1}\alpha\right) \\
&\qquad+\left(1-\delta\left(\frac{1}{2}\right)\right)^s\varphi\left(\frac{\varsigma}{\lambda+1}\alpha+\frac{\lambda+1-\varsigma}{\lambda+1}\beta\right) \\
&\leq \delta^s\left(\frac{1}{2}\right)\left[\delta^s\left(\frac{\varsigma}{\lambda+1}\right)\varphi(\beta)+m\left(1-\delta\left(\frac{\varsigma}{\lambda+1}\right)\right)^s\varphi\left(\frac{\alpha}{m}\right)\right] \\
&\qquad+\left(1-\delta\left(\frac{1}{2}\right)\right)^s\left[\delta^s\left(\frac{\varsigma}{\lambda+1}\right)\varphi(\alpha)+m\left(1-\delta\left(\frac{\varsigma}{\lambda+1}\right)\right)^s\varphi\left(\frac{\beta}{m}\right)\right].
\end{aligned}$$

Multiplying by $w''(\varsigma)$, integrating over $[0,1]$ and collecting terms provide the second bound in (4). This completes the proof. ☐

Remark 5. *Corollary 17 coincides with Theorem 1 of* [13].

Remark 6. *Setting $w''(\varsigma)=1$, $\lambda=0$, $m=s=1$ and $\delta(\varsigma)=\varsigma$ in* (4) *yields the classical Hermite-Hadamard inequality.*

Remark 7. *Under the same configuration as in Remark 19, replacing $w''(\varsigma)$ by $\frac{\exp(-\gamma\varsigma)}{\eta}$ with $\gamma=\frac{1-\eta}{\eta}(\beta-\alpha)$, leads to Theorem 1 of* [2].

Remark 8. *Choosing $w'(\varsigma)=\frac{\varsigma^\eta}{\eta}$ together with $\lambda=0$, $m=s=1$ and $\delta(\varsigma)=\varsigma$ transforms* (4) *into an expression involving Riemann-Liouville fractional integrals. In this case, one retrieves Theorem 2 in* [27]*:*

$$\varphi\left(\frac{\alpha+\beta}{2}\right)\leq\frac{\Gamma(\eta+1)}{2(\beta-\alpha)^\eta}\left[I^\eta_{\beta-}\varphi(\alpha)+I^\eta_{\alpha+}\varphi(\beta)\right]\leq\frac{\varphi(\alpha)+\varphi(\beta)}{2}.$$

Remark 9. *For $w'(\varsigma)=\frac{\varsigma^\eta}{\eta}$ and $\lambda=m=s=1$, relation* (4) *reduces to Theorem 4 in* [28]*. Furthermore, when restricted to s-convex functions with $\lambda=0$, it reproduces the main result of Theorem 2.1 in* [30]*. Additional related statements are discussed in Theorem 3 of* [29] *for $w'(\varsigma)=\varsigma^\eta$. Under analogous assumptions, results for m-convex functions, including Theorem 5 of the present work, follow in a direct manner.*

Remark 10. *Several results formulated in terms of k-Riemann–Liouville fractional integrals, such as Theorem 5 in* [31] *(see also Theorem 1 in* [24]*), can be derived from Corollary 17 by selecting $w'(\varsigma)=\varsigma^{\frac{\eta}{k}}$, $k>0$, $\lambda=0$, $m=s=1$ and $\delta(\varsigma)=\varsigma$.*

Remark 11. *Let $\eta\in[0,1]$, $\lambda=0$, $m=s=1$, and $\delta(\varsigma)=\varsigma$ in* (4)*, which corresponds to the convex framework. Assume further that*

$$w''(\varsigma)=\frac{\eta\varsigma^{\frac{\eta}{k}-1}}{kN(\eta)\Gamma_k(\eta)},$$

with $k>0$. Under these assumptions, the left-hand side of (4) *takes the form:*

$$\begin{aligned}
\varphi\left(\frac{\alpha+\beta}{2}\right)\frac{1}{N(\eta)\Gamma_k(\eta)}\leq\frac{1}{2(\beta-\alpha)^{\frac{\eta}{k}}}&\left[\frac{\eta}{kN(\eta)\Gamma_k(\eta)}\int_\alpha^\beta\frac{\varphi(u)}{(u-\alpha)^{1-\frac{\eta}{k}}}\,du\right. \\
&\left.+\frac{\eta}{kN(\eta)\Gamma_k(\eta)}\int_\alpha^\beta\frac{\varphi(u)}{(\beta-u)^{1-\frac{\eta}{k}}}\,du\right].
\end{aligned}$$

This relation can equivalently be written as:

$$\varphi\left(\frac{\alpha+\beta}{2}\right)\frac{2(\beta-\alpha)^{\frac{\eta}{k}}}{N(\eta)\Gamma_k(\eta)} \leq \frac{\eta}{kN(\eta)\Gamma_k(\eta)}\int_\alpha^\beta \frac{\varphi(u)}{(u-\alpha)^{1-\frac{\eta}{k}}}\,du + \frac{\eta}{kN(\eta)\Gamma_k(\eta)}\int_\alpha^\beta \frac{\varphi(u)}{(\beta-u)^{1-\frac{\eta}{k}}}\,du. \tag{7}$$

Adding the quantity $\frac{1-\eta}{N(\eta)}(\varphi(\alpha)+\varphi(\beta))$ *to both sides of* (7) *and using the estimate*

$$\varphi\left(\frac{\alpha+\beta}{2}\right) \leq \frac{\varphi(\alpha)+\varphi(\beta)}{2},$$

one arrives at:

$$\varphi\left(\frac{\alpha+\beta}{2}\right)\left[\frac{2(\beta-\alpha)^{\frac{\eta}{k}}}{N(\eta)\Gamma_k(\eta)} + \frac{(1-\eta)}{N(\eta)}\right] \leq^{AB} I^\eta_{\alpha+}\varphi(\beta) +^{AB} I^\eta_{\beta-}\varphi(\alpha). \tag{8}$$

A comparable argument applied to the right-hand side of (1) *yields:*

$$\begin{aligned}&\frac{1}{2(\beta-\alpha)^{\frac{\eta}{k}}}\left[\frac{\eta}{kN(\eta)\Gamma_k(\eta)}\int_\alpha^\beta \frac{\varphi(u)}{(u-\alpha)^{1-\frac{\eta}{k}}}\,du + \frac{\eta}{kN(\eta)\Gamma_k(\eta)}\int_\alpha^\beta \frac{\varphi(u)}{(\beta-u)^{1-\frac{\eta}{k}}}\,du\right]\\ \leq\ & \frac{1}{N(\eta)\Gamma_k(\eta)}\left(\frac{\varphi(\alpha)+\varphi(\beta)}{2}\right).\end{aligned}$$

After multiplying by $2(\beta-\alpha)^{\frac{\eta}{k}}$*, adding* $\frac{(1-\eta)}{N(\eta)}(\varphi(\alpha)+\varphi(\beta))$ *and simplifying, we obtain:*

$${}^{AB}I^\eta_{\alpha+}\varphi(\beta) +^{AB} I^\eta_{\beta-}\varphi(\alpha) \leq \left[\frac{2(\beta-\alpha)^{\frac{\eta}{k}}}{N(\eta)\Gamma_k(\eta)} + \frac{(1-\eta)}{N(\eta)}\right]\left(\frac{\varphi(\alpha)+\varphi(\beta)}{2}\right). \tag{9}$$

Together (8) *and* (9) *produce a bound closely related to Theorem 6 in* [18]. *In particular, the choice* $k=1$ *leads to a formulation comparable with Proposition 2.1 in* [12].

Corollary 2. *By fixing* $m=1$, $\delta(\varsigma)=\varsigma$, $\lambda=0$, $\eta\in[0,1]$ *and* $w''(\varsigma)=1$ *in* (4)*, one obtains:*

$$2^{s-1}\varphi\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_\alpha^\beta \varphi(\varsigma)\,d\varsigma \leq \frac{\varphi(\alpha)+\varphi(\beta)}{s+1}.$$

Remark 12. *Corollary 25 is equivalent to Theorem 1.5 of* [23].

Remark 13. *By putting* $s=1$ *in Corollary 25, one recovers Theorem 2 of* [14] *after straightforward algebraic manipulations.*

Remark 14. *For every* $t\in(\alpha,\beta)$*, the classical integral mean can be represented in terms of the left and right Caputo-Fabrizio fractional integrals (see Definition 9) as follows:*

$$\begin{aligned}\frac{1}{\beta-\alpha}\int_\alpha^\beta \varphi(\varsigma)\,d\varsigma &= \frac{N(\eta)}{\eta(\beta-\alpha)}\left[\frac{1-\eta}{N(\eta)}\varphi(t) + \frac{\eta}{N(\eta)}\int_\alpha^t \varphi(\varsigma)\,d\varsigma\right.\\ &\qquad \left. + \frac{1-\eta}{N(\eta)}\varphi(t) + \frac{\eta}{N(\eta)}\int_t^\beta \varphi(\varsigma)\,d\varsigma - \frac{2(1-\eta)}{N(\eta)}\varphi(t)\right]\\ &= \frac{N(\eta)}{\eta(\beta-\alpha)}\left[\left({}^{CF}_{\alpha}I^\eta\varphi\right)(t) + \left({}^{CF}I^\eta_\beta\varphi\right)(t) - \frac{2(1-\eta)}{N(\eta)}\varphi(t)\right].\end{aligned}$$

Consequently, Theorem 2.1 of [23] *is a direct outcome of combining this representation with Corollary 25.*

Remark 15. *Taking $\varphi(\varsigma) = (f * g)(\varsigma)$ in (4), a new formulation for convolution functions emerges:*

$$
\begin{aligned}
&(f * g)\left(\tfrac{\alpha+\beta}{2}\right)(w'(1) - w'(0)) \\
&\leq \frac{\lambda+1}{\beta-\alpha}\left[h^s\left(\tfrac{1}{2}\right) J^w_{\alpha^+}(f * g)\left(\tfrac{\beta+\lambda\alpha}{\lambda+1}\right) + \left(1 - h\left(\tfrac{1}{2}\right)\right)^s J^w_{\beta^-}(f * g)\left(\tfrac{\alpha+\lambda\beta}{\lambda+1}\right)\right] \\
&\leq \left[\delta^s\left(\tfrac{1}{2}\right)(f * g)(\beta) + \left(1 - \delta\left(\tfrac{1}{2}\right)\right)^s (f * g)(\alpha)\right]\left[\int_0^1 w''(\varsigma)\,\delta^s\left(\tfrac{\varsigma}{\lambda+1}\right) d\varsigma\right] \\
&\quad + m\left[\delta^s\left(\tfrac{1}{2}\right)(f * g)\left(\tfrac{\alpha}{m}\right) + \left(1 - \delta\left(\tfrac{1}{2}\right)\right)^s (f * g)\left(\tfrac{\beta}{m}\right)\right]\left[\int_0^1 w''(\varsigma)\left(1 - \delta\left(\tfrac{\varsigma}{\lambda+1}\right)\right)^s d\varsigma\right].
\end{aligned}
$$

The previous remarks highlight the versatility of the main result and indicate how further estimates associated with alternative fractional operators may be obtained.

3|Generalizations

In the rest of the article, to simplify the notation of some mathematical expressions, we will adopt the following notations:

$$
\begin{aligned}
\mathbf{T}_1 \;:=\;& w(1)\left[\Lambda_{\beta,\kappa}(1) + \Lambda_{\alpha,\kappa}(1)\right] - w(0)\left[\Lambda_{\beta,\kappa}(0) + \Lambda_{\alpha,\kappa}(0)\right] \\
&- w(1)\left[\varphi'\left(\Lambda_{\alpha,\kappa}(1)\right)\tfrac{\partial \Lambda_{\alpha,\kappa}}{\partial\varsigma}(1) + \varphi'\left(\Lambda_{\beta,\kappa}(1)\right)\tfrac{\partial \Lambda_{\beta,\kappa}}{\partial\varsigma}(1)\right] \\
&+ w(0)\left[\varphi'\left(\Lambda_{\alpha,\kappa}(0)\right)\tfrac{\partial \Lambda_{\alpha,\kappa}}{\partial\varsigma}(0) + \varphi'\left(\Lambda_{\beta,\kappa}(0)\right)\tfrac{\partial \Lambda_{\beta,\kappa}}{\partial\varsigma}(0)\right] \\
&- \mathcal{L}^w_{\varphi}\Lambda_{\beta,\kappa}(\varsigma) - \mathcal{L}^w_{\varphi}\Lambda_{\alpha,\kappa}(\varsigma)
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\mathbf{T}_2 \;:=\;& w(1)\left[\left(\frac{\beta-\kappa}{\lambda+1}\right)\varphi'(\beta) - \left(\frac{\alpha-\kappa}{\lambda+1}\right)\varphi'(\alpha)\right] \\
&+ w(0)\left[\left(\frac{\alpha-\kappa}{\lambda+1}\right)\varphi'\left(\frac{\lambda\alpha+\kappa}{\lambda+1}\right) - \left(\frac{\beta-\kappa}{\lambda+1}\right)\varphi'\left(\frac{\lambda\beta+\kappa}{\lambda+1}\right)\right] \\
&- w'(1)\left[\varphi(\alpha) + \varphi(\beta)\right] + w'(0)\left[\varphi\left(\frac{\lambda\alpha+\kappa}{\lambda+1}\right) + \varphi\left(\frac{\lambda\beta+\kappa}{\lambda+1}\right)\right] \\
&+ \left(\frac{\lambda+1}{\kappa-\alpha}\right) J^w_{\alpha^+}\varphi\left(\frac{\lambda\alpha+\kappa}{\lambda+1}\right) + \left(\frac{\lambda+1}{\beta-\kappa}\right) J^w_{\beta^-}\varphi\left(\frac{\lambda\beta+\kappa}{\lambda+1}\right)
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\mathbf{T}_3 : \;=\;& w(1)\left[\left(\frac{\beta-\kappa}{\lambda+1}\right)(f * g)'(\beta) - \left(\frac{\alpha-\kappa}{\lambda+1}\right)(f * g)'(\alpha)\right] \\
&+ w(0)\left[\left(\frac{\alpha-\kappa}{\lambda+1}\right)(f * g)'\left(\frac{\lambda\alpha+\kappa}{\lambda+1}\right) - \left(\frac{\beta-\kappa}{\lambda+1}\right)(f * g)'\left(\frac{\lambda\beta+\kappa}{\lambda+1}\right)\right] \\
&- w'(1)\left[(f * g)(\alpha) + \varphi(\beta)\right] + w'(0)\left[(f * g)\left(\frac{\lambda\alpha+\kappa}{\lambda+1}\right) + (f * g)\left(\frac{\lambda\beta+\kappa}{\lambda+1}\right)\right] \\
&+ \left(\frac{\lambda+1}{\kappa-\alpha}\right) J^w_{\alpha^+}(f * g)\left(\frac{\lambda\alpha+\kappa}{\lambda+1}\right) + \left(\frac{\lambda+1}{\beta-\kappa}\right) J^w_{\beta^-}(f * g)\left(\frac{\lambda\beta+\kappa}{\lambda+1}\right)
\end{aligned} \tag{12}
$$

The subsequent identity will serve as a fundamental tool moving forward.

Lemma 1. *Let be a function φ defined over a closed interval $D \subset \mathbb{R}$, which it is twice differentiable in the interior D^o. Consider a weight function $w : [0,1] \to \mathbb{R}$ and $\Lambda(\varsigma, \vartheta, \rho) = \Phi(\varsigma)\vartheta + \Psi(\varsigma)\rho$ a weighted path with $\vartheta, \rho \in [\alpha, \beta] \subset D^o$. Provided that φ'' belongs to $L[\alpha, \beta]$, the subsequent equality can be established:*

$$\begin{aligned}
\mathbf{T}_1 &= \int_0^1 w(\varsigma)\varphi''\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\left(\tfrac{\partial\Lambda_{\alpha,\kappa}}{\partial\varsigma}(\varsigma)\right)^2 d\varsigma + \mathbf{E}_1 \\
&\quad + \int_0^1 w(\varsigma)\varphi''\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\left(\tfrac{\partial\Lambda_{\beta,\kappa}}{\partial\varsigma}(\varsigma)\right)^2 d\varsigma + \mathbf{E}_2, \qquad (13)
\end{aligned}$$

for $\mathbf{T}_1$ defined in (10) and $\kappa \in [\alpha, \beta]$. Moreover,

$$\mathbf{E}_1 = \int_0^1 w(\varsigma)\varphi'\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\tfrac{\partial^2\Lambda_{\alpha,\kappa}}{\partial\varsigma^2}(\varsigma)\, d\varsigma,$$
$$\mathbf{E}_2 = \int_0^1 w(\varsigma)\varphi'\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\tfrac{\partial^2\Lambda_{\beta,\kappa}}{\partial\varsigma^2}(\varsigma)\, d\varsigma.$$

Proof: Taking into account the Definition (12), we find:

$$\mathcal{L}_\varphi^w\Lambda_{\alpha,\kappa}(\varsigma) + \mathcal{L}_\varphi^w\Lambda_{\beta,\kappa}(\varsigma) = \int_0^1 w''(\varsigma)\varphi\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\, d\varsigma + \int_0^1 w''(\varsigma)\varphi\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\, d\varsigma = (\mathcal{I}_1) + (\mathcal{I}_2) \qquad (14)$$

By applying integration by parts twice, we obtain:

$$\begin{aligned}
(\mathcal{I}_1) &= w'(1)\varphi\left(\Lambda_{\alpha,\kappa}(1)\right) - w'(0)\varphi\left(\Lambda_{\alpha,\kappa}(0)\right) - \int_0^1 w'(\varsigma)\varphi'\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\tfrac{\partial\Lambda_{\alpha,\kappa}}{\partial\varsigma}(\varsigma)\, d\varsigma \\
&= w'(1)\varphi\left(\Lambda_{\alpha,\kappa}(1)\right) - w'(0)\varphi\left(\Lambda_{\alpha,\kappa}(0)\right) \\
&\quad -\Bigg\{ w(1)\varphi'\left(\Lambda_{\alpha,\kappa}(1)\right)\tfrac{\partial\Lambda_{\alpha,\kappa}}{\partial\varsigma}(1) - w(0)\varphi'\left(\Lambda_{\alpha,\kappa}(0)\right)\tfrac{\partial\Lambda_{\alpha,\kappa}}{\partial\varsigma}(0) \\
&\qquad - \int_0^1 w(\varsigma)\Big[\varphi''\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\left(\tfrac{\partial\Lambda_{\alpha,\kappa}}{\partial\varsigma}(\varsigma)\right)^2 + \varphi'\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\tfrac{\partial^2\Lambda_{\alpha,\kappa}}{\partial\varsigma^2}(\varsigma)\Big]\, d\varsigma\Bigg\}. \qquad (15)
\end{aligned}$$

In a manner analogous to the above,

$$\begin{aligned}
(\mathcal{I}_2) &= w'(1)\varphi\left(\Lambda_{\beta,\kappa}(1)\right) - w'(0)\varphi\left(\Lambda_{\beta,\kappa}(0)\right) \\
&\quad -\Bigg\{ w(1)\varphi'\left(\Lambda_{\beta,\kappa}(1)\right)\tfrac{\partial\Lambda_{\beta,\kappa}}{\partial\varsigma}(1) - w(0)\varphi'\left(\Lambda_{\beta,\kappa}(0)\right)\tfrac{\partial\Lambda_{\beta,\kappa}}{\partial\varsigma}(0) \\
&\qquad - \int_0^1 w(\varsigma)\Big[\varphi''\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\left(\tfrac{\partial\Lambda_{\beta,\kappa}}{\partial\varsigma}(\varsigma)\right)^2 + \varphi'\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\tfrac{\partial^2\Lambda_{\beta,\kappa}}{\partial\varsigma^2}(\varsigma)\Big]\, d\varsigma\Bigg\}. \qquad (16)
\end{aligned}$$

Equation (13) emerges once (14)–(16) are considered together and their terms are rearranged. □

Corollary 3. *Let the assumptions of Lemma 30 be in force. We set*

$$\Lambda_{\alpha,\kappa}(\varsigma) = \frac{\varsigma}{\lambda+1}\,\alpha + \frac{\lambda+1-\varsigma}{\lambda+1}\,\kappa,$$
$$\Lambda_{\beta,\kappa}(\varsigma) = \frac{\varsigma}{\lambda+1}\,\beta + \frac{\lambda+1-\varsigma}{\lambda+1}\,\kappa,$$

where $\lambda \in \mathbb{N} \cup \{0\}$. With this specification, the identity below follows:

$$\mathbf{T}_2 = \left(\frac{\alpha-\kappa}{\lambda+1}\right)^2 \int_0^1 w(\varsigma)\varphi''\left(\frac{\varsigma}{\lambda+1}\alpha + \frac{\lambda+1-\varsigma}{\lambda+1}\kappa\right) d\varsigma + \left(\frac{\beta-\kappa}{\lambda+1}\right)^2 \int_0^1 w(\varsigma)\varphi''\left(\frac{\varsigma}{\lambda+1}\beta + \frac{\lambda+1-\varsigma}{\lambda+1}\kappa\right) d\varsigma. \tag{17}$$

with $\mathbf{T}_2$ *introduced in* (11) *and* $\kappa \in [\alpha, \beta]$.

Proof: First we denote

$$(\mathcal{I}_1) = \left(\frac{\alpha-\kappa}{\lambda+1}\right)^2 \int_0^1 w(\varsigma)\varphi''\left(\frac{\varsigma}{\lambda+1}\alpha + \frac{\lambda+1-\varsigma}{\lambda+1}\kappa\right) d\varsigma.$$
$$(\mathcal{I}_2) = \left(\frac{\beta-\kappa}{\lambda+1}\right)^2 \int_0^1 w(\varsigma)\varphi''\left(\frac{\varsigma}{\lambda+1}\beta + \frac{\lambda+1-\varsigma}{\lambda+1}\kappa\right) d\varsigma.$$

We apply integration by parts twice to $(\mathcal{I}_1)$ together with the concept of weighted fractional integrals introduced in [9], giving:

$$\begin{aligned}
(\mathcal{I}_1) &= -\left(\frac{\alpha-\kappa}{\lambda+1}\right)\left[w(1)\varphi'(\alpha) - w(0)\varphi'\left(\frac{\lambda\alpha+\kappa}{\lambda+1}\right)\right] \\
&\quad -\left[w'(1)\varphi(\alpha) - w'(0)\varphi\left(\frac{\lambda\alpha+\kappa}{\lambda+1}\right)\right] \\
&\quad +\int_0^1 w''(\varsigma)\varphi\left(\frac{\varsigma}{\lambda+1}\alpha + \frac{\lambda+1-\varsigma}{\lambda+1}\kappa\right) d\varsigma \\
&= -\left(\frac{\alpha-\kappa}{\lambda+1}\right)\left[w(1)\varphi'(\alpha) - w(0)\varphi'\left(\frac{\lambda\alpha+\kappa}{\lambda+1}\right)\right] \\
&\quad -\left[w'(1)\varphi(\alpha) - w'(0)\varphi\left(\frac{\lambda\alpha+\kappa}{\lambda+1}\right)\right] \\
&\quad +\left(\frac{\lambda+1}{\kappa-\alpha}\right)\int_\alpha^{\left(\frac{\lambda\alpha+\kappa}{\lambda+1}\right)} w''\left(\frac{\frac{\lambda\alpha+\kappa}{\lambda+1}-\mu}{\frac{\lambda\alpha+\kappa}{\lambda+1}-\alpha}\right)\varphi(\mu)\, d\mu \\
&= -\left(\frac{\alpha-\kappa}{\lambda+1}\right)\left[w(1)\varphi'(\alpha) - w(0)\varphi'\left(\frac{\lambda\alpha+\kappa}{\lambda+1}\right)\right] \\
&\quad -\left[w'(1)\varphi(\alpha) - w'(0)\varphi\left(\frac{\lambda\alpha+\kappa}{\lambda+1}\right)\right] \\
&\quad +\left(\frac{\lambda+1}{\kappa-\alpha}\right) J^w_{\alpha^+}\varphi\left(\frac{\lambda\alpha+\kappa}{\lambda+1}\right).
\end{aligned}$$

In a manner similar to the above but for $(\mathcal{I}_2)$,

$$\begin{aligned}(\mathcal{I}_2) &= \left(\frac{\beta-\kappa}{\lambda+1}\right)\left[w(1)\varphi'(\beta)-w(0)\varphi'\left(\frac{\lambda\beta+\kappa}{\lambda+1}\right)\right] \\ &\quad -\left[w'(1)\varphi(\beta)-w'(0)\varphi\left(\frac{\lambda\beta+\kappa}{\lambda+1}\right)\right] \\ &\quad +\left(\frac{\lambda+1}{\beta-\kappa}\right)J^{w}_{\beta-}\varphi\left(\frac{\lambda\beta+\kappa}{\lambda+1}\right). \end{aligned} \tag{18}$$

Adding (18) with (18) and conveniently grouping the terms, it follows (17). □

Remark 16. *Corollary 31 coincides with Lemma 2.1 of* [22].

Corollary 4. *Assume that the hypotheses of Lemma 30 are satisfied. Consider the weight function*

$$w(\varsigma)=\varsigma(1-\varsigma)$$

together with the weighted path

$$\Lambda_{\alpha,\beta}(\varsigma)=\varsigma\alpha+(1-\varsigma)\beta.$$

for $\varsigma\in(0,1)$. *Then*

$$\frac{\varphi(\alpha)+\varphi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta\varphi(z)\,dz=\frac{(\beta-\alpha)^2}{2}\int_0^1\varsigma(1-\varsigma)\varphi''(\varsigma\alpha+(1-\varsigma)\beta)\;d\varsigma. \tag{19}$$

Proof: By applying the change of variable $z=\varsigma\alpha+(1-\varsigma)\beta$, the integral on the right-hand side of (19) becomes:

$$\int_0^1\varsigma(1-\varsigma)\varphi''(\varsigma\alpha+(1-\varsigma)\beta)\,d\varsigma=\frac{1}{(\beta-\alpha)^3}\int_\alpha^\beta(z-\beta)(\alpha-z)\varphi''(z)\,dz. \tag{20}$$

Integrating by parts in (20), yields:

$$\begin{aligned}\int_\alpha^\beta(z-\beta)(\alpha-z)\varphi''(z)\,dz &= \left[(z-\beta)(\alpha-z)\varphi'(z)\right]_\alpha^\beta-\int_\alpha^\beta(\alpha+\beta-2z)\varphi'(z)\,dz \\ &= -\int_\alpha^\beta(\alpha+\beta-2z)\varphi'(z)\,dz.\end{aligned}$$

A second integration by parts, produces:

$$\begin{aligned}-\int_\alpha^\beta(\alpha+\beta-2z)\varphi'(z)\,dz &= -\left[(\alpha+\beta-2z)\varphi(z)\right]_\alpha^\beta+2\int_\alpha^\beta\varphi(z)\,dz \\ &= (\beta-\alpha)(\varphi(\alpha)+\varphi(\beta))+2\int_\alpha^\beta\varphi(z)\,dz.\end{aligned} \tag{21}$$

Combining (21) and (22), we obtain:

$$\int_\alpha^\beta(z-\beta)(\alpha-z)\varphi''(z)\,dz=(\beta-\alpha)(\varphi(\alpha)+\varphi(\beta))+2\int_\alpha^\beta\varphi(z)\,dz. \tag{22}$$

Substituting (23) into (20) gives:

$$\begin{aligned}\int_0^1 \varsigma(1-\varsigma)\varphi''(\varsigma\alpha+(1-\varsigma)\beta)\,d\varsigma &= \frac{1}{(\beta-\alpha)^3}\left[(\beta-\alpha)(\varphi(\alpha)+\varphi(\beta))+2\int_\alpha^\beta \varphi(z)\,dz\right]\\ &= \frac{\varphi(\alpha)+\varphi(\beta)}{(\beta-\alpha)^2}+\frac{2}{(\beta-\alpha)^3}\int_\alpha^\beta \varphi(z)\,dz. \qquad (23)\end{aligned}$$

Finally, multiplying both sides of (23) by $\frac{(\beta-\alpha)^2}{2}$ leads to (19). □

Remark 17. *The identity established in Corollary 33 is exactly the one appearing as Lemma 1 in* [26].

3.1|Results on (Λ,δ)-Convexity of the First Type

Theorem 2. *Let be a function φ defined over a closed interval $D\subset\mathbb{R}$, which it is twice differentiable in the interior D^o. Consider a weight function $w:[0,1]\to\mathbb{R}$ and $\Lambda(\varsigma,\vartheta,\rho)=\Phi(\varsigma)\vartheta+\Psi(\varsigma)\rho$ a weighted path with $\vartheta,\rho\in[\alpha,\beta]\subset D^o$. Suppose that $|\varphi''|$ satisfies (Λ,δ)- convexity condition of the first type over $\left[\alpha,\frac{\beta}{m}\right]\subset D^o$ and $m\in[0,1]$. Then it holds that:*

$$\begin{aligned}|\mathbf{T}_1| \quad &\leq \quad \left[(\mathcal{M}_1)^2|\varphi''(\alpha)|+(\mathcal{M}_2)^2|\varphi''(\beta)|\right]\int_0^1 w(\varsigma)\Phi(\delta(\varsigma))\,d\varsigma \qquad (24)\\ &+m\left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|\left[(\mathcal{M}_1)^2+(\mathcal{M}_2)^2\right]\int_0^1 w(\varsigma)\Psi(\delta(\varsigma))\,d\varsigma+|\mathbf{E}_1|+|\mathbf{E}_2|,\end{aligned}$$

where $\kappa\in[\alpha,\beta]$ and $\mathbf{T}_1$ is defined in (10). Moreover,

$$\mathcal{M}_1=\max_{\varsigma\in[0,1]}\left|\frac{\partial\Lambda_{\alpha,\kappa}}{\partial\varsigma}(\varsigma)\right|,\quad \mathcal{M}_2=\max_{\varsigma\in[0,1]}\left|\frac{\partial\Lambda_{\beta,\kappa}}{\partial\varsigma}(\varsigma)\right|.$$

Proof: We proceed by applying Lemma (30) together with the triangle inequality, we have:

$$\begin{aligned}|\mathbf{T}_1| &\leq \int_0^1 w(\varsigma)\left|\varphi''\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\right|\left|\tfrac{\partial\Lambda_{\alpha,\kappa}}{\partial\varsigma}\right|^2 d\varsigma+|\mathbf{E}_1|\\ &\quad+\int_0^1 w(\varsigma)\left|\varphi''\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\right|\left|\tfrac{\partial\Lambda_{\beta,\kappa}}{\partial\varsigma}\right|^2 d\varsigma+|\mathbf{E}_2|\\ &\leq(\mathcal{M}_1)^2\int_0^1 w(\varsigma)\left|\varphi''\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\right|\,d\varsigma+|\mathbf{E}_1|\\ &\quad+(\mathcal{M}_2)^2\int_0^1 w(\varsigma)\left|\varphi''\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\right|\,d\varsigma+|\mathbf{E}_2|. \qquad (25)\end{aligned}$$

Employing the first-type (Λ,δ)- convexity property of $|\varphi''|$, we find:

$$\begin{aligned}&(\mathcal{M}_1)^2\int_0^1 w(\varsigma)\left|\varphi''\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\right|\,d\varsigma \qquad (26)\\ &\leq(\mathcal{M}_1)^2\int_0^1 w(\varsigma)\Big[\Phi(\delta(\varsigma))\left|\varphi''(\alpha)\right|+m\Psi(\delta(\varsigma))\left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|\Big]\,d\varsigma\\ &=(\mathcal{M}_1)^2\left[\left|\varphi''(\alpha)\right|\int_0^1 w(\varsigma)\Phi(\delta(\varsigma))\,d\varsigma+m\left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|\int_0^1 w(\varsigma)\Psi(\delta(\varsigma))\,d\varsigma\right].\end{aligned}$$

Analogously,

$$
\begin{aligned}
& (\mathcal{M}_2)^2 \int_0^1 w(\varsigma) \left|\varphi''\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\right| d\varsigma \\
\leq\ & (\mathcal{M}_2)^2 \Big[\left|\varphi''(\beta)\right| \int_0^1 w(\varsigma)\Phi(\delta(\varsigma))\, d\varsigma + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right| \int_0^1 w(\varsigma)\Psi(\delta(\varsigma))\, d\varsigma \Big]. \qquad (27)
\end{aligned}
$$

By assembling the relations (25)–(27) and suitably rearranging the expressions, we infer (24). □

Theorem 3. *Let be a function φ defined over a closed interval $D \subset \mathbb{R}$, which it is twice differentiable in the interior D^o. Consider a weight function $w : [0,1] \to \mathbb{R}$ and $\Lambda(\varsigma, \vartheta, \rho) = \Phi(\varsigma)\vartheta + \Psi(\varsigma)\rho$ a weighted path with $\vartheta, \rho \in [\alpha, \beta] \subset D^o$. Suppose that $|\varphi''|^q$ satisfies (Λ, δ)- convexity condition of the first type over $\left[\alpha, \frac{\beta}{m}\right] \subset D^o$ and $m \in [0,1]$. Then this implies that:*

$$
\begin{aligned}
|\mathbf{T}_1| \leq\ & \left(\int_0^1 w^p(\varsigma)\, d\varsigma\right)^{\frac{1}{p}} \Bigg[(\mathcal{M}_1)^2 \left(|\varphi''(\alpha)|^q \int_0^1 \Phi(\delta(\varsigma))\, d\varsigma + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \Psi(\delta(\varsigma))\, d\varsigma \right)^{\frac{1}{q}} \\
& + (\mathcal{M}_2)^2 \left(|\varphi''(\beta)|^q \int_0^1 \Phi(\delta(\varsigma))\, d\varsigma + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \Psi(\delta(\varsigma))\, d\varsigma \right)^{\frac{1}{q}} \Bigg] + |\mathbf{E}_1| + |\mathbf{E}_2|,
\end{aligned}
$$

where $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\kappa \in [\alpha, \beta]$ and $\mathbf{T}_1$ is defined in (10).

Proof: We know that:

$$
\begin{aligned}
|\mathbf{T}_1| \leq\ & (\mathcal{M}_1)^2 \int_0^1 w(\varsigma) \left|\varphi''\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\right| d\varsigma + |\mathbf{E}_1| \\
& + (\mathcal{M}_2)^2 \int_0^1 w(\varsigma) \left|\varphi''\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\right| d\varsigma + |\mathbf{E}_2|. \qquad (28)
\end{aligned}
$$

By means of Hölder's inequality applied to (28), one obtains:

$$
\begin{aligned}
|\mathbf{T}_1| \leq\ & \left(\int_0^1 w^p(\varsigma)\, d\varsigma\right)^{\frac{1}{p}} \Bigg[(\mathcal{M}_1)^2 \left(\int_0^1 \left|\varphi''\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\right|^q dt \right)^{\frac{1}{q}} \\
& + (\mathcal{M}_2)^2 \left(\int_0^1 \left|\varphi''\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\right|^q dt \right)^{\frac{1}{q}} \Bigg] + |\mathbf{E}_1| + |\mathbf{E}_2|, \qquad (29)
\end{aligned}
$$

for $\frac{1}{p} + \frac{1}{q} = 1$ and $p, q > 1$.

Utilizing the first-type (Λ, δ)- convexity property of $|\varphi''|^q$, we refine (29) as follows:

$$
\begin{aligned}
|\mathbf{T}_1| \leq\ & \left(\int_0^1 w^p(\varsigma)\, d\varsigma\right)^{\frac{1}{p}} \Bigg[(\mathcal{M}_1)^2 \left(|\varphi''(\alpha)|^q \int_0^1 \Phi(\delta(\varsigma))\, d\varsigma + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \Psi(\delta(\varsigma))\, d\varsigma \right)^{\frac{1}{q}} \\
& + (\mathcal{M}_2)^2 \left(|\varphi''(\beta)|^q \int_0^1 \Phi(\delta(\varsigma))\, d\varsigma + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \Psi(\delta(\varsigma))\, d\varsigma \right)^{\frac{1}{q}} \Bigg] + |\mathbf{E}_1| + |\mathbf{E}_2|.
\end{aligned}
$$

Thus, the verification is completed. □

Theorem 4. *Let be a function φ defined over a closed interval $D \subset \mathbb{R}$, which it is twice differentiable in the interior D^o. Consider a weight function $w : [0,1] \to \mathbb{R}$ and $\Lambda(\varsigma, \vartheta, \rho) = \Phi(\varsigma)\vartheta + \Psi(\varsigma)\rho$ a weighted path with $\vartheta, \rho \in [\alpha, \beta] \subset D^o$. Suppose that $|\varphi'|$ satisfies (Λ, δ)- convexity condition of the first type over $\left[\alpha, \frac{\beta}{m}\right] \subset D^o$ and $m \in [0,1]$. Then the result below follows:*

$$
\begin{aligned}
|\mathbf{T}_1| \leq \left(\int_0^1 w(\varsigma)\, d\varsigma\right)^{1-\frac{1}{q}} \Bigg[(\mathcal{M}_1)^2 \Bigg(& |\varphi''(\alpha)|^q \int_0^1 w(\varsigma)\, \Phi\big(\delta(\varsigma)\big)\, d\varsigma \\
& + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 w(\varsigma)\, \Psi\big(\delta(\varsigma)\big)\, d\varsigma \Bigg)^{\frac{1}{q}} \\
+ (\mathcal{M}_2)^2 \Bigg(& |\varphi''(\beta)|^q \int_0^1 w(\varsigma)\, \Phi\big(\delta(\varsigma)\big)\, d\varsigma \\
& + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 w(\varsigma)\, \Psi\big(\delta(\varsigma)\big)\, d\varsigma \Bigg)^{\frac{1}{q}} \Bigg] + |\mathbf{E}_1| + |\mathbf{E}_2|,
\end{aligned}
\tag{30}
$$

where $q \geq 1$, $\kappa \in [\alpha, \beta]$ and $\mathbf{T}_1$ is defined in (10).

Proof: We know that:

$$
\begin{aligned}
|\mathbf{T}_1| \quad \leq \quad & (\mathcal{M}_1)^2 \int_0^1 w(\varsigma)\, |\varphi''(\Lambda_{\alpha,\kappa}(\varsigma))|\, d\varsigma + |\mathbf{E}_1| \\
& + (\mathcal{M}_2)^2 \int_0^1 w(\varsigma)\, |\varphi''(\Lambda_{\beta,\kappa}(\varsigma))|\, d\varsigma + |\mathbf{E}_2|.
\end{aligned}
\tag{31}
$$

Employing the power mean inequality in (32), we get:

$$
\begin{aligned}
|\mathbf{T}_1| \quad \leq \quad & \left(\int_0^1 w(\varsigma)\, d\varsigma\right)^{1-\frac{1}{q}} \left[(\mathcal{M}_1)^2 \left(\int_0^1 w(\varsigma)\, |\varphi''(\Lambda_{\alpha,\kappa}(\varsigma))|^q\, d\varsigma\right)^{\frac{1}{q}} \right. \\
& \left. + (\mathcal{M}_2)^2 \left(\int_0^1 w(\varsigma)\, |\varphi''(\Lambda_{\beta,\kappa}(\varsigma))|^q\, d\varsigma\right)^{\frac{1}{q}} \right] + |\mathbf{E}_1| + |\mathbf{E}_2|,
\end{aligned}
\tag{32}
$$

with $q \geq 1$.

We utilize the first-type (Λ, δ)- convexity property of $|\varphi''|^q$ in (32), obtaining:

$$
\begin{aligned}
|\mathbf{T}_1| \leq \left(\int_0^1 w(\varsigma)\, d\varsigma\right)^{1-\frac{1}{q}} \Bigg[(\mathcal{M}_1)^2 \Bigg(& |\varphi''(\alpha)|^q \int_0^1 w(\varsigma)\, \Phi\big(\delta(\varsigma)\big)\, d\varsigma \\
& + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 w(\varsigma)\, \Psi\big(\delta(\varsigma)\big)\, d\varsigma \Bigg)^{\frac{1}{q}} \\
+ (\mathcal{M}_2)^2 \Bigg(& |\varphi''(\beta)|^q \int_0^1 w(\varsigma)\, \Phi\big(\delta(\varsigma)\big)\, d\varsigma \\
& + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 w(\varsigma)\, \Psi\big(\delta(\varsigma)\big)\, d\varsigma \Bigg)^{\frac{1}{q}} \Bigg] + |\mathbf{E}_1| + |\mathbf{E}_2|.
\end{aligned}
$$

The assertion is thus verified. □

Theorem 5. *Let be a function φ defined over a closed interval $D \subset \mathbb{R}$, which it is twice differentiable in the interior D^o. Consider a weight function $w : [0,1] \to \mathbb{R}$ and $\Lambda(\varsigma, \vartheta, \rho) = \Phi(\varsigma)\vartheta + \Psi(\varsigma)\rho$ a weighted path with $\vartheta, \rho \in [\alpha, \beta] \subset D^o$. Suppose that $|\varphi''|^q$ satisfies (Λ, δ)- convexity condition of the first type over $\left[\alpha, \frac{\beta}{m}\right] \subset D^o$ and $m \in [0,1]$. Then this we leads to:*

$$\begin{aligned} |\mathbf{T}_1| \leq\ & \mathcal{M}_1\Bigg[\frac{1}{p}\left(\int_0^1 w^p(\varsigma)\,d\varsigma\right)^{\frac{1}{p}} + \frac{1}{q}\Bigg(|\varphi''(\alpha)|^q \int_0^1 \Phi(\delta(\varsigma))\,d\varsigma \\ & + m\left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \Psi(\delta(\varsigma))\,d\varsigma\Bigg)^{\frac{1}{q}}\Bigg] \\ & + \mathcal{M}_2\Bigg[\frac{1}{p}\left(\int_0^1 w^p(\varsigma)\,d\varsigma\right)^{\frac{1}{p}} + \frac{1}{q}\Bigg(|\varphi''(\beta)|^q \int_0^1 \Phi(\delta(\varsigma))\,d\varsigma \\ & + m\left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \Psi(\delta(\varsigma))\,d\varsigma\Bigg)^{\frac{1}{q}}\Bigg] + |\mathbf{E}_1| + |\mathbf{E}_2|, \end{aligned}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, $\kappa \in [\alpha, \beta]$ and $\mathbf{T}_1$ is defined in (10).

Proof: We know that:

$$\begin{aligned} |\mathbf{T}_1| \leq\ & (\mathcal{M}_1)^2 \int_0^1 w(\varsigma)\left|\varphi''\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\right| d\varsigma + |\mathbf{E}_1| \\ & + (\mathcal{M}_2)^2 \int_0^1 w(\varsigma)\left|\varphi''\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\right| d\varsigma + |\mathbf{E}_2|. \end{aligned} \tag{33}$$

Employing Young's inequality in (33), we obtain:

$$\begin{aligned} |\mathbf{T}_1| \leq\ & (\mathcal{M}_1)^2\left[\frac{1}{p}\left(\int_0^1 w^p(\varsigma)\,d\varsigma\right)^{\frac{1}{p}} + \frac{1}{q}\left(\int_0^1 \left|\varphi''\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\right|^q dt\right)^{\frac{1}{q}}\right] + |\mathbf{E}_1| \\ & + (\mathcal{M}_2)^2\left[\frac{1}{p}\left(\int_0^1 w^p(\varsigma)\,d\varsigma\right)^{\frac{1}{p}} + \frac{1}{q}\left(\int_0^1 \left|\varphi''\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\right|^q dt\right)^{\frac{1}{q}}\right] + |\mathbf{E}_2|, \end{aligned} \tag{34}$$

for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Utilizing the first-type (Λ, δ)- convexity property of $|\varphi'|^q$, we refine (34) as follows:

$$\begin{aligned} |\mathbf{T}_1| \leq\ & \mathcal{M}_1\Bigg[\frac{1}{p}\left(\int_0^1 w^p(\varsigma)\,d\varsigma\right)^{\frac{1}{p}} + \frac{1}{q}\Bigg(|\varphi''(\alpha)|^q \int_0^1 \Phi(\delta(\varsigma))\,d\varsigma \\ & + m\left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \Psi(\delta(\varsigma))\,d\varsigma\Bigg)^{\frac{1}{q}}\Bigg] \\ & + \mathcal{M}_2\Bigg[\frac{1}{p}\left(\int_0^1 w^p(\varsigma)\,d\varsigma\right)^{\frac{1}{p}} + \frac{1}{q}\Bigg(|\varphi''(\beta)|^q \int_0^1 \Phi(\delta(\varsigma))\,d\varsigma \\ & + m\left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \Psi(\delta(\varsigma))\,d\varsigma\Bigg)^{\frac{1}{q}}\Bigg] + |\mathbf{E}_1| + |\mathbf{E}_2|. \end{aligned}$$

Which completes the proof. □

3.2|Results on (Λ, δ)-Convexity of the Second Type

Theorem 6. *Let be a function φ defined over a closed interval $D \subset \mathbb{R}$, which it is twice differentiable in the interior D^o. Consider a weight function $w : [0,1] \to \mathbb{R}$ and $\Lambda(\varsigma, \vartheta, \rho) = \Phi(\varsigma)\vartheta + \Psi(\varsigma)\rho$ a weighted path with $\vartheta, \rho \in [\alpha, \beta] \subset D^o$. Suppose that $|\varphi''|$ satisfies (Λ, δ)- convexity condition of the second type over $\left[\alpha, \frac{\beta}{m}\right] \subset D^o$ and $m \in [0,1]$. Then the subsequent relation is valid:*

$$
\begin{aligned}
|\mathbf{T}_1| \quad \leq \quad & \Big[(\mathcal{M}_1)^2 \, |\varphi''(\alpha)| + (\mathcal{M}_2)^2 \, |\varphi''(\beta)| \Big] \int_0^1 w(\varsigma)\Phi(\delta^s(\varsigma)) \, d\varsigma \\
& + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right| \Big[(\mathcal{M}_1)^2 + (\mathcal{M}_2)^2 \Big] \int_0^1 w(\varsigma) \left(\Psi(\delta(\varsigma))\right)^s \, d\varsigma + |\mathbf{E}_1| + |\mathbf{E}_2|,
\end{aligned} \tag{35}
$$

for $\kappa \in [\alpha, \beta]$ and $\mathbf{T}_1$ defined in (10).

Proof: By following a strategy akin to that employed in the Theorem 35, we get:

$$
\begin{aligned}
|\mathbf{T}_1| \quad \leq \quad & (\mathcal{M}_1)^2 \int_0^1 w(\varsigma) \left|\varphi''\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\right| d\varsigma + |\mathbf{E}_1| \\
& + (\mathcal{M}_2)^2 \int_0^1 w(\varsigma) \left|\varphi''\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\right| d\varsigma + |\mathbf{E}_2|.
\end{aligned} \tag{36}
$$

By means of second-type (Λ, δ)- convexity property of $|\varphi''|$ applied to (36), one derives:

$$
\begin{aligned}
& (\mathcal{M}_1)^2 \int_0^1 w(\varsigma) \left|\varphi''\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\right| d\varsigma \\
\leq \quad & (\mathcal{M}_1)^2 \int_0^1 w(\varsigma) \Big[\Phi(\delta(\varsigma)) \left|\varphi''(\alpha)\right| + m\Psi(\delta(\varsigma)) \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right| \Big] d\varsigma \\
= \quad & (\mathcal{M}_1)^2 \Big[\left|\varphi''(\alpha)\right| \int_0^1 w(\varsigma)\Phi(\delta^s(\varsigma)) \, d\varsigma + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right| \int_0^1 w(\varsigma) \left(\Psi(\delta(\varsigma))\right)^s \, d\varsigma \Big].
\end{aligned} \tag{37}
$$

In an analogous way,

$$
\begin{aligned}
& (\mathcal{M}_2)^2 \int_0^1 w(\varsigma) \left|\varphi''\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\right| d\varsigma \\
\leq \quad & (\mathcal{M}_2)^2 \Big[\left|\varphi''(\beta)\right| \int_0^1 w(\varsigma)\Phi(\delta(\varsigma)) \, d\varsigma + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right| \int_0^1 w(\varsigma)\Psi(\delta(\varsigma)) \, d\varsigma \Big].
\end{aligned} \tag{38}
$$

By combining (36) - (38) and conveniently grouping the terms, it follows (35). □

Corollary 5. *Under the assumptions of Theorem 39 and suppose additionally that $|\varphi''|$ satisfies the modified (h, m)-convexity condition of the second type. Then, it's true that:*

$$
\begin{aligned}
|\mathbf{T}_2| \quad \leq \quad & \left[\left(\frac{\alpha - \kappa}{\lambda + 1} \right)^2 |\varphi''(\alpha)| + \left(\frac{\beta - \kappa}{\lambda + 1} \right)^2 |\varphi''(\beta)| \right] \int_0^1 w(\varsigma) \, \delta^s \left(\frac{\lambda + \varsigma}{\lambda + 1} \right) \, d\varsigma \\
& + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right| \left[\frac{(\alpha - \kappa)^2 + (\beta - \kappa)^2}{\lambda + 1} \right] \int_0^1 w(\varsigma) \left(1 - \delta \left(\frac{\lambda + \varsigma}{\lambda + 1} \right) \right)^s \, d\varsigma.
\end{aligned} \tag{39}
$$

with $\mathbf{T}_2$ introduced in (11).

Proof: By employing Corollary 31 together with the triangle inequality, we get:

$$
\begin{aligned}
|\mathbf{T}_2| \quad \leq \quad & \left|\frac{\alpha-\kappa}{\lambda+1}\right|^2 \int_0^1 w(\varsigma)\left|\varphi''\left(\frac{\lambda+\varsigma}{\lambda+1}\alpha+\frac{1-\varsigma}{\lambda+1}\kappa\right)\right| d\varsigma \\
& +\left|\frac{\beta-\kappa}{\lambda+1}\right|^2 \int_0^1 w(\varsigma)\left|\varphi''\left(\frac{\lambda+\varsigma}{\lambda+1}\beta+\frac{1-\varsigma}{\lambda+1}\kappa\right)\right| d\varsigma.
\end{aligned}
$$

We utilize (h, m)-convexity of $|\varphi''|$, deducing:

$$
\begin{aligned}
& \left|\frac{\alpha-\kappa}{\lambda+1}\right|^2 \int_0^1 w(\varsigma)\left|\varphi''\left(\frac{\lambda+\varsigma}{\lambda+1}\alpha+\frac{1-\varsigma}{\lambda+1}\kappa\right)\right| d\varsigma \\
\leq \quad & \left|\frac{\alpha-\kappa}{\lambda+1}\right|^2 \int_0^1 w(\varsigma)\left[\delta^s\left(\frac{\lambda+\varsigma}{\lambda+1}\right)|\varphi''(\alpha)|+m\left(1-\delta\left(\frac{\lambda+\varsigma}{\lambda+1}\right)\right)^s|\varphi''(\kappa)|\right] d\varsigma \\
= \quad & \left|\frac{\alpha-\kappa}{\lambda+1}\right|^2\left[|\varphi''(\alpha)| \int_0^1 w(\varsigma)\, \delta^s\left(\frac{\lambda+\varsigma}{\lambda+1}\right) d\varsigma\right. \\
& \left. +m\left|\varphi''\left(\tfrac{\kappa}{m}\right)\right| \int_0^1 w(\varsigma)\left(1-\delta\left(\frac{\lambda+\varsigma}{\lambda+1}\right)\right)^s d\varsigma\right]. \qquad (40)
\end{aligned}
$$

In a manner similar to the above,

$$
\begin{aligned}
& \left|\frac{\beta-\kappa}{\lambda+1}\right|^2 \int_0^1 w(\varsigma)\left|\varphi''\left(\frac{\lambda+\varsigma}{\lambda+1}\alpha+\frac{1-\varsigma}{\lambda+1}\beta\right)\right| d\varsigma \\
\leq \quad & \left|\frac{\beta-\kappa}{\lambda+1}\right|^2\left[|\varphi''(\alpha)| \int_0^1 w(\varsigma)\, \delta^s\left(\frac{\lambda+\varsigma}{\lambda+1}\right) d\varsigma\right. \\
& \left. +m\left|\varphi''\left(\frac{\beta}{m}\right)\right| \int_0^1 w(\varsigma)\left(1-\delta\left(\frac{\lambda+\varsigma}{\lambda+1}\right)\right)^s d\varsigma\right]. \qquad (41)
\end{aligned}
$$

Combining (40) and (41) and simplifying the resulting expression gives (39). ☐

Remark 18. *Corollary 40 is equivalent to Theorem 2.11 of* [22].

Remark 19. *Replacing $\varphi(\varsigma)=(f*g)(\varsigma)$ in* (39)*, an alternative representation for convolution-type functions is obtained:*

$$
\begin{aligned}
|\mathbf{T}_3| \quad \leq \quad & \left[\left(\frac{\alpha-\kappa}{\lambda+1}\right)^2|(f*g)''(\alpha)|+\left(\frac{\beta-\kappa}{\lambda+1}\right)^2|(f*g)''(\beta)|\right] \int_0^1 w(\varsigma)\, \delta^s\left(\frac{\lambda+\varsigma}{\lambda+1}\right) d\varsigma \\
& +m\left|(f*g)''\left(\tfrac{\kappa}{m}\right)\right|\left[\frac{(\alpha-\kappa)^2+(\beta-\kappa)^2}{\lambda+1}\right] \int_0^1 w(\varsigma)\left(1-\delta\left(\frac{\lambda+\varsigma}{\lambda+1}\right)\right)^s d\varsigma.
\end{aligned}
$$

where $\mathbf{T}_3$ is given by (12).

Corollary 6. *Under the assumptions of Theorem 39, suppose further that there exists a constant* $\mathrm{M}>0$ *such that* $|\varphi''(z)| \leq \mathrm{M}$ *for every* $z \in[\alpha, \beta]$*. Consider the particular weight function*

$$w(\varsigma)=\varsigma(1-\varsigma),$$

and the linear weighted path

$$\Lambda_{\alpha, \beta}(\varsigma)=\varsigma \alpha+(1-\varsigma) \beta,$$

for $\varsigma \in(0,1)$.

Then

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\varphi(z)\,dz\right|\leq\frac{\pi}{32}\,\mathrm{M}\,(\beta-\alpha)^2.$$

Proof: Invoking Corollary 33 and Definition 4 for $s=1$, yields:

$$\begin{aligned}\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\varphi(z)\,dz\right| &\leq \frac{(\beta-\alpha)^2}{2}\int_0^1 \varsigma(1-\varsigma)\left|\varphi''\big(\varsigma\alpha+(1-\varsigma)\beta\big)\right|\,d\varsigma\\ &\leq \frac{(\beta-\alpha)^2}{2}\int_0^1 \varsigma(1-\varsigma)\left[\varsigma|\varphi''(\alpha)|+(1-\varsigma)|\varphi''(\beta)|\right]\,d\varsigma.\end{aligned}\tag{42}$$

Since $\varsigma\in(0,1)$, we have the elementary bounds:

$$\varsigma\leq\frac{\sqrt{\varsigma}}{2\sqrt{1-\varsigma}},\qquad 1-\varsigma\leq\frac{\sqrt{1-\varsigma}}{2\sqrt{\varsigma}}.\tag{43}$$

From (42) and (43), we get:

$$\begin{aligned}\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\varphi(z)\,dz\right| &\leq \frac{(\beta-\alpha)^2}{4}\left[\left(\int_0^1 \varsigma^{3/2}(1-\varsigma)^{1/2}\,d\varsigma\right)|\varphi''(\alpha)|\right.\\ &\qquad\left.+\left(\int_0^1 \varsigma^{1/2}(1-\varsigma)^{3/2}\,d\varsigma\right)|\varphi''(\beta)|\right].\end{aligned}$$

The two integrals involved are equal:

$$\int_0^1 \varsigma^{3/2}(1-\varsigma)^{1/2}\,d\varsigma=\int_0^1 \varsigma^{1/2}(1-\varsigma)^{3/2}\,d\varsigma=\frac{\pi}{16}.\tag{44}$$

Replacing (44) in (43), one obtains:

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\varphi(z)\,dz\right|\leq\frac{\pi}{64}(\beta-\alpha)^2\big(|\varphi''(\alpha)|+|\varphi''(\beta)|\big).\tag{45}$$

Finally, the boundedness hypothesis of $|\varphi''|$ leads to:

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\varphi(z)\,dz\right|\leq\frac{\pi}{32}\mathrm{M}\,(\beta-\alpha)^2.$$

The proof is now complete. □

Remark 20. *Corollary 43 is identical to Theorem 2.1 of* [26].

Theorem 7. *Let be a function φ defined over a closed interval $D\subset\mathbb{R}$, which it is twice differentiable in the interior D^o. Consider a weight function $w:[0,1]\to\mathbb{R}$ and $\Lambda(\varsigma,\vartheta,\rho)=\Phi(\varsigma)\vartheta+\Psi(\varsigma)\rho$ a weighted path with $\vartheta,\rho\in[\alpha,\beta]\subset D^o$. Suppose that $|\varphi''|^q$ satisfies the (Λ,δ)- convexity condition of second type over $\left[\alpha,\frac{\beta}{m}\right]\subset D^o$, the following bound is valid:*

$$\begin{aligned}|\mathbf{T}_1| &\leq \left(\int_0^1 w^p(\varsigma)\,d\varsigma\right)^{\frac{1}{p}}\left[(\mathcal{M}_1)^2\left(|\varphi''(\alpha)|^q\int_0^1\Phi(\delta^s(\varsigma))d\varsigma + m\left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q\int_0^1(\Psi(\delta(\varsigma)))^s\,d\varsigma\right)^{\frac{1}{q}}\right.\\ &\left.+(\mathcal{M}_2)^2\left(|\varphi''(\beta)|^q\int_0^1\Phi(\delta^s(\varsigma))d\varsigma + m\left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q\int_0^1(\Psi(\delta(\varsigma)))^s\,d\varsigma\right)^{\frac{1}{q}}\right] + |\mathbf{E}_1| + |\mathbf{E}_2|,\end{aligned}$$

for $\kappa\in[\alpha,\beta]$ *and* $\mathbf{T}_1$ *defined in (10).*

Proof: We know that:

$$\begin{aligned}|\mathbf{T}_1| &\leq (\mathcal{M}_1)^2\int_0^1 w(\varsigma)\left|\varphi''\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\right|d\varsigma + |\mathbf{E}_1|\\ &+(\mathcal{M}_2)^2\int_0^1 w(\varsigma)\left|\varphi''\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\right|d\varsigma + |\mathbf{E}_2|.\end{aligned}\tag{46}$$

Employing Hölder's inequality in (46), we get:

$$\begin{aligned}|\mathbf{T}_1| &\leq \left(\int_0^1 w^p(\varsigma)\,d\varsigma\right)^{\frac{1}{p}}\left[(\mathcal{M}_1)^2\left(\int_0^1\left|\varphi''\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\ &\left.+(\mathcal{M}_2)^2\left(\int_0^1\left|\varphi''\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\right|^q dt\right)^{\frac{1}{q}}\right] + |\mathbf{E}_1| + |\mathbf{E}_2|,\end{aligned}\tag{47}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $p,q>1$.

We utilize the second-type (Λ,δ)- convexity property of $|\varphi'|^q$ in (47), obtaining:

$$\begin{aligned}|\mathbf{T}_1| &\leq \left(\int_0^1 w^p(\varsigma)\,d\varsigma\right)^{\frac{1}{p}}\left[(\mathcal{M}_1)^2\left(|\varphi''(\alpha)|^q\int_0^1\Phi(\delta^s(\varsigma))d\varsigma + m\left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q\int_0^1(\Psi(\delta(\varsigma)))^s\,d\varsigma\right)^{\frac{1}{q}}\right.\\ &\left.+(\mathcal{M}_2)^2\left(|\varphi''(\beta)|^q\int_0^1\Phi(\delta^s(\varsigma))d\varsigma + m\left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q\int_0^1(\Psi(\delta(\varsigma)))^s\,d\varsigma\right)^{\frac{1}{q}}\right] + |\mathbf{E}_1| + |\mathbf{E}_2|.\end{aligned}$$

The desired result is verificated. □

Corollary 7. *Adopting the hypotheses of Theorem 39, suppose in addition that* $|\varphi''|^q$ *belongs to the class of modified* (h,m)*-convex functions of the second type. Then it's implies that:*

$$\begin{aligned}|\mathbf{T}_2| \leq &\left(\int_0^1 w^p(\varsigma)\,d\varsigma\right)^{\frac{1}{p}}\left[\left(\frac{\alpha-\kappa}{\lambda+1}\right)^2\left(|\varphi''(\alpha)|^q\int_0^1\delta^s\left(\frac{\lambda+\varsigma}{\lambda+1}\right)d\varsigma\right.\right.\\ &\left.+\left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q\int_0^1\left(1-\delta\left(\frac{\lambda+\varsigma}{\lambda+1}\right)\right)^s d\varsigma\right)^{\frac{1}{q}}\\ &\left.+\left(\frac{\beta-\kappa}{\lambda+1}\right)^2\left(|\varphi''(\beta)|^q\int_0^1\delta^s\left(\frac{\lambda+\varsigma}{\lambda+1}\right)d\varsigma+\left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q\int_0^1\left(1-\delta\left(\frac{\lambda+\varsigma}{\lambda+1}\right)\right)^s d\varsigma\right)^{\frac{1}{q}}\right],\end{aligned}\tag{48}$$

with $\mathbf{T}_2$ *introduced in* (11).

Proof: Invoking the estimation proved earlier:

$$\begin{aligned}|\mathbf{T}_2| \quad &\leq \quad \left|\frac{\alpha-\kappa}{\lambda+1}\right|^2 \int_0^1 w(\varsigma)\left|\varphi''\left(\frac{\lambda+\varsigma}{\lambda+1}\alpha+\frac{1-\varsigma}{\lambda+1}\kappa\right)\right| d\varsigma \\ &\quad + \left|\frac{\beta-\kappa}{\lambda+1}\right|^2 \int_0^1 w(\varsigma)\left|\varphi''\left(\frac{\lambda+\varsigma}{\lambda+1}\beta+\frac{1-\varsigma}{\lambda+1}\kappa\right)\right| d\varsigma. \end{aligned} \tag{49}$$

We apply Hölder's inequality to (49), deriving:

$$\begin{aligned}|\mathbf{T}_2| \leq \left(\int_0^1 w^p(\varsigma)\, d\varsigma\right)^{\frac{1}{p}} &\left[\left(\frac{\alpha-\kappa}{\lambda+1}\right)^2 \left(\int_0^1 \left|\varphi''\left(\frac{\lambda+\varsigma}{\lambda+1}\alpha+\frac{1-\varsigma}{\lambda+1}\kappa\right)\right|^q d\varsigma\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\frac{\beta-\kappa}{\lambda+1}\right)^2 \left(\int_0^1 \left|\varphi''\left(\frac{\lambda+\varsigma}{\lambda+1}\beta+\frac{1-\varsigma}{\lambda+1}\kappa\right)\right|^q d\varsigma\right)^{\frac{1}{q}}\right], \end{aligned} \tag{50}$$

for $\frac{1}{p}+\frac{1}{q}=1$ and $p,q>1$.

Utilizing the (h,m)-convexity of $|\varphi''|^q$ in (50), the desired bound follows. □

Remark 21. *Corollary 46 coincides with Theorem 2.14 of* [22].

Remark 22. *Substituting $\varphi(\varsigma)=(f*g)(\varsigma)$ into (48) yields a convolution-based formulation:*

$$\begin{aligned}|\mathbf{T}_3| \quad \leq \quad &\left(\int_0^1 w^p(\varsigma)\, d\varsigma\right)^{\frac{1}{p}} \left[\left(\frac{\alpha-\kappa}{\lambda+1}\right)^2 \left(|(f*g)''(\alpha)|^q \int_0^1 \delta^s\left(\frac{\lambda+\varsigma}{\lambda+1}\right) d\varsigma\right.\right. \\ &\left.+\left|(f*g)''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \left(1-\delta\left(\frac{\lambda+\varsigma}{\lambda+1}\right)\right)^s d\varsigma\right)^{\frac{1}{q}} \\ &+\left(\frac{\beta-\kappa}{\lambda+1}\right)^2 \left(|(f*g)''(\beta)|^q \int_0^1 \delta^s\left(\frac{\lambda+\varsigma}{\lambda+1}\right) d\varsigma\right. \\ &\left.\left.+\left|(f*g)''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \left(1-\delta\left(\frac{\lambda+\varsigma}{\lambda+1}\right)\right)^s d\varsigma\right)^{\frac{1}{q}}\right]. \end{aligned}$$

where $\mathbf{T}_3$ is given by (12).

Corollary 8. *Suppose that all the assumptions of Theorem 39 hold. In addition, assume additionally that there exists a constant $\mathrm{M}>0$ such that $|\varphi''(z)|\leq \mathrm{M}$ for every $z\in[\alpha,\beta]$. Let*

$$w(\varsigma)=\varsigma(1-\varsigma),$$

and

$$\Lambda_{\alpha,\beta}(\varsigma)=\varsigma\alpha+(1-\varsigma)\beta,$$

for $\varsigma\in(0,1)$.

Then

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta \varphi(z)\, dz\right| \leq \frac{(\beta-\alpha)^2}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{q}} M\left[\mathrm{B}\left(p+1,p+1\right)\right]^{\frac{1}{p}},$$

where $\mathrm{B}(\cdot,\cdot)$ denotes the Euler beta function.

Proof: Invoking Corollary 33 and then applying Hölder's inequality, we arrive at:

$$\begin{aligned}\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta \varphi(z)\,dz\right| &\le \frac{(\beta-\alpha)^2}{2}\left[\int_0^1 \varsigma^p(1-\varsigma)^p\,d\varsigma\right]^{\frac{1}{p}} \\ &\quad\times\left[\int_0^1 |\varphi''(\varsigma\alpha+(1-\varsigma)\beta)|^q\,d\varsigma\right]^{\frac{1}{q}}. \qquad (51)\end{aligned}$$

By Definition 4, the second-type (Λ,δ)-convexity for $s=1$ implies:

$$\left|\varphi''\big(\varsigma\alpha+(1-\varsigma)\beta\big)\right| \le \varsigma|\varphi''(\alpha)|+(1-\varsigma)|\varphi''(\beta)|. \qquad (52)$$

Moreover, the elementary inequalities

$$\varsigma \le \frac{\sqrt{\varsigma}}{2\sqrt{1-\varsigma}}, \qquad 1-\varsigma \le \frac{\sqrt{1-\varsigma}}{2\sqrt{\varsigma}}, \qquad (53)$$

are valid for all $\varsigma \in (0,1)$.

Combining (51) - (53), we derive:

$$\begin{aligned}\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta \varphi(z)\,dz\right| &\le \frac{(\beta-\alpha)^2}{2}\left[\mathrm{B}(p+1,p+1)\right]^{\frac{1}{p}} \\ &\quad\times\left[\frac{1}{2}\int_0^1 \frac{\sqrt{\varsigma}}{\sqrt{1-\varsigma}}\,d\varsigma\,|\varphi''(\alpha)|^q\right. \\ &\quad\left.+\frac{1}{2}\int_0^1 \frac{\sqrt{1-\varsigma}}{\sqrt{\varsigma}}\,d\varsigma\,|\varphi''(\beta)|^q\right]^{\frac{1}{q}}. \qquad (54)\end{aligned}$$

Using the identities

$$\int_0^1 \frac{\sqrt{\varsigma}}{\sqrt{1-\varsigma}}\,d\varsigma = \int_0^1 \frac{\sqrt{1-\varsigma}}{\sqrt{\varsigma}}\,d\varsigma = \frac{\pi}{2},$$

the expression (54) becomes:

$$\begin{aligned}\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta \varphi(z)\,dz\right| &\le \tfrac{(\beta-\alpha)^2}{2}\left(\tfrac{\pi}{4}\right)^{\frac{1}{q}}\left[\mathrm{B}(p+1,p+1)\right]^{\frac{1}{p}} \\ &\quad\times\left[|\varphi''(\alpha)|^q+|\varphi''(\beta)|^q\right]^{\frac{1}{q}}. \qquad (55)\end{aligned}$$

Utilizing the fact that $|\varphi''(z)| \le \mathrm{M}$ for $z \in [\alpha,\beta]$ in (55), it follows that:

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta \varphi(z)\,dz\right| \le \frac{(\beta-\alpha)^2}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{q}}\mathrm{M}\left[\mathrm{B}\left(p+1,p+1\right)\right]^{\frac{1}{p}}.$$

This completes the proof. □

Remark 23. *Corollary 49 is precisely Theorem 2.1 of* [26].

Corollary 9. *Under the framework of Corollary 49, the following estimate is likewise true:*

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\varphi(z)\,dz\right|\leq\frac{(\beta-\alpha)^2}{2^{1+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}M\left[\mathrm{B}\left(\frac{1}{2},q+\frac{1}{2}\right)\right]^{\frac{1}{q}},$$

where $\mathrm{B}(\cdot,\cdot)$ *denotes the Euler beta function.*

Remark 24. *Corollary 51 reproduces the same statement as Theorem 2.3 in* [26].

Theorem 8. *Let* φ *be a twice differentiable function defined on a closed interval* $D\subset\mathbb{R}$*. Consider a weight function* $w:[0,1]\to\mathbb{R}$ *and let* $\Lambda(\varsigma,\vartheta,\rho)=\Phi(\varsigma)\vartheta+\Psi(\varsigma)\rho$ *be a weighted path with* $\vartheta,\rho\in[\alpha,\beta]\subset D^{\circ}$*. Suppose that* $|\varphi''|^q$ *satisfies the* (Λ,δ)*-convexity condition of the second type on* $\left[\alpha,\frac{\beta}{m}\right]\subset D^{\circ}$*. Then the following estimate holds:*

$$\begin{aligned}|\mathbf{T}_1|\leq\left(\int_0^1 w(\varsigma)\,d\varsigma\right)^{1-\frac{1}{q}}&\Bigg[(\mathcal{M}_1)^2\Bigg(|\varphi''(\alpha)|^q\int_0^1 w(\varsigma)\,\Phi\big(\delta^s(\varsigma)\big)\,d\varsigma\\&\quad+m\left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q\int_0^1 w(\varsigma)\,\big(\Psi\big(\delta(\varsigma)\big)\big)^s\,d\varsigma\Bigg)^{\frac{1}{q}}\\&+(\mathcal{M}_2)^2\Bigg(|\varphi''(\beta)|^q\int_0^1 w(\varsigma)\,\Phi\big(\delta^s(\varsigma)\big)\,d\varsigma\\&\quad+m\left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q\int_0^1 w(\varsigma)\,\big(\Psi\big(\delta(\varsigma)\big)\big)^s\,d\varsigma\Bigg)^{\frac{1}{q}}\Bigg]+|\mathbf{E}_1|+|\mathbf{E}_2|,\end{aligned}$$

for $\kappa\in[\alpha,\beta]$ *and* $\mathbf{T}_1$ *defined in (10).*

Proof: We know that:

$$\begin{aligned}|\mathbf{T}_1|\quad\leq\quad&(\mathcal{M}_1)^2\int_0^1 w(\varsigma)\,|\varphi''(\Lambda_{\alpha,\kappa}(\varsigma))|\,d\varsigma+|\mathbf{E}_1|\\&+(\mathcal{M}_2)^2\int_0^1 w(\varsigma)\,|\varphi''(\Lambda_{\beta,\kappa}(\varsigma))|\,d\varsigma+|\mathbf{E}_2|.\end{aligned}\tag{56}$$

By means of power mean inequality applied to (57), one derives:

$$\begin{aligned}|\mathbf{T}_1|\quad\leq\quad&\left(\int_0^1 w(\varsigma)\,d\varsigma\right)^{1-\frac{1}{q}}\left[(\mathcal{M}_1)^2\left(\int_0^1 w(\varsigma)\,|\varphi''(\Lambda_{\alpha,\kappa}(\varsigma))|^q\,d\varsigma\right)^{\frac{1}{q}}\right.\\&\left.+(\mathcal{M}_2)^2\left(\int_0^1 w(\varsigma)\,|\varphi''(\Lambda_{\beta,\kappa}(\varsigma))|^q\,d\varsigma\right)^{\frac{1}{q}}\right]+|\mathbf{E}_1|+|\mathbf{E}_2|,\end{aligned}\tag{57}$$

with $q\geq 1$.

Utilizing the second-type (Λ,δ)- convexity property of $|\varphi'|^q$ in (57), it holds that:

$$
\begin{aligned}
|\mathbf{T}_1| \leq \left(\int_0^1 w(\varsigma)\, d\varsigma\right)^{1-\frac{1}{q}} \Bigg[(\mathcal{M}_1)^2 \Bigg(|\varphi''(\alpha)|^q \int_0^1 w(\varsigma)\, \Phi\big(\delta^s(\varsigma)\big)\, d\varsigma \\
+ m \left|\varphi''\Big(\frac{\kappa}{m}\Big)\right|^q \int_0^1 w(\varsigma) \left(\Psi(\delta(\varsigma))\right)^s d\varsigma \Bigg)^{\frac{1}{q}} \\
+ (\mathcal{M}_2)^2 \Bigg(|\varphi''(\beta)|^q \int_0^1 w(\varsigma)\, \Phi\big(\delta^s(\varsigma)\big)\, d\varsigma \\
+ m \left|\varphi''\Big(\frac{\kappa}{m}\Big)\right|^q \int_0^1 w(\varsigma) \left(\Psi(\delta(\varsigma))\right)^s d\varsigma \Bigg)^{\frac{1}{q}} \Bigg] + |\mathbf{E}_1| + |\mathbf{E}_2|.
\end{aligned}
$$

Which was to be proved. □

Corollary 10. *Let the hypotheses of Theorem 39 hold. If, in addition, $|\varphi''|^q$ is modified (h, m)-convex of the second type, then $\mathbf{T}_2$ satisfies:*

$$
\begin{aligned}
|\mathbf{T}_2| \leq \; & \left(\int_0^1 w(\varsigma)\, d\varsigma\right)^{1-\frac{1}{q}} \Bigg[\left(\frac{\alpha-\kappa}{\lambda+1}\right)^2 \Bigg(|\varphi''(\alpha)|^q \int_0^1 \delta^s\left(\frac{\lambda+\varsigma}{\lambda+1}\right) d\varsigma \\
& + |\varphi''(\tfrac{\kappa}{m})|^q \int_0^1 \left(1-\delta\left(\frac{\lambda+\varsigma}{\lambda+1}\right)\right)^s d\varsigma \Bigg)^{\frac{1}{q}} \\
& + \left(\frac{\beta-\kappa}{\lambda+1}\right)^2 \Bigg(|\varphi''(\beta)|^q \int_0^1 \delta^s\left(\frac{\lambda+\varsigma}{\lambda+1}\right) d\varsigma \\
& + |\varphi''(\tfrac{\kappa}{m})|^q \int_0^1 \left(1-\delta\left(\frac{\lambda+\varsigma}{\lambda+1}\right)\right)^s d\varsigma \Bigg)^{\frac{1}{q}} \Bigg].
\end{aligned} \tag{58}
$$

with $\mathbf{T}_2$ introduced in (11).

Proof: From the known expression:

$$
\begin{aligned}
|\mathbf{T}_2| \leq \; & \left|\frac{\alpha-\kappa}{\lambda+1}\right|^2 \int_0^1 w(\varsigma) \left|\varphi''\left(\frac{\lambda+\varsigma}{\lambda+1}\alpha + \frac{1-\varsigma}{\lambda+1}\kappa\right)\right| d\varsigma \\
& + \left|\frac{\beta-\kappa}{\lambda+1}\right|^2 \int_0^1 w(\varsigma) \left|\varphi''\left(\frac{\lambda+\varsigma}{\lambda+1}\beta + \frac{1-\varsigma}{\lambda+1}\kappa\right)\right| d\varsigma.
\end{aligned} \tag{59}
$$

We apply the mean power inequality to (59), inferring:

$$
\begin{aligned}
|\mathbf{T}_2| \leq \left(\int_0^1 w(\varsigma)\, d\varsigma\right)^{1-\frac{1}{q}} \Bigg[\left(\frac{\alpha-\kappa}{\lambda+1}\right)^2 \left(\int_0^1 \left|\varphi''\left(\frac{\lambda+\varsigma}{\lambda+1}\alpha + \frac{1-\varsigma}{\lambda+1}\kappa\right)\right|^q d\varsigma\right)^{\frac{1}{q}} \\
+ \left(\frac{\beta-\kappa}{\lambda+1}\right)^2 \left(\int_0^1 \left|\varphi''\left(\frac{\lambda+\varsigma}{\lambda+1}\beta + \frac{1-\varsigma}{\lambda+1}\kappa\right)\right|^q d\varsigma\right)^{\frac{1}{q}} \Bigg],
\end{aligned} \tag{60}
$$

for $q \geq 1$.

Utilizing the (h, m)-convexity of $|\varphi''|^q$ in (60), the desired bound follows. □

Remark 25. *Corollary 54 coincides with Theorem 2.17 of* [22].

Remark 26. *Setting $\varphi(\varsigma) = (f * g)(\varsigma)$ in (58) yields a new convolution-based formulation:*

$$\begin{aligned}|\mathbf{T}_3| \leq & \left(\int_0^1 w(\varsigma)\,d\varsigma\right)^{1-\frac{1}{q}} \left[\left(\frac{\alpha-\kappa}{\lambda+1}\right)^2 \left(|(f*g)''(\alpha)|^q \int_0^1 \delta^s\left(\frac{\lambda+\varsigma}{\lambda+1}\right) d\varsigma \right.\right. \\ & \left. + \left|(f*g)''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \left(1-\delta\left(\frac{\lambda+\varsigma}{\lambda+1}\right)\right)^s d\varsigma \right)^{\frac{1}{q}} \\ & + \left(\frac{\beta-\kappa}{\lambda+1}\right)^2 \left(|(f*g)''(\beta)|^q \int_0^1 \delta^s\left(\frac{\lambda+\varsigma}{\lambda+1}\right) d\varsigma \right. \\ & \left.\left. + \left|(f*g)''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \left(1-\delta\left(\frac{\lambda+\varsigma}{\lambda+1}\right)\right)^s d\varsigma \right)^{\frac{1}{q}} \right].\end{aligned}$$

where $\mathbf{T}_3$ is given by (12)*.*

Corollary 11. *Assume that the hypotheses of Theorem 39 are fulfilled and $|\varphi''(z)| \leq \mathrm{M}$ for all $z \in [\alpha, \beta]$. Consider*

$$w(\varsigma) = \varsigma(1-\varsigma),$$

and

$$\Lambda_{\alpha,\beta}(\varsigma) = \varsigma\alpha + (1-\varsigma)\beta,$$

for $\varsigma \in (0,1)$.

Then

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta \varphi(z)\,dz\right| \leq \frac{(\beta-\alpha)^2}{4}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{\pi}{8}\right)^{\frac{1}{q}}\mathrm{M}.$$

Proof: Invoking Corollary 33 and then applying power mean inequality, one results:

$$\begin{aligned}\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta \varphi(z)\,dz\right| \leq & \frac{(\beta-\alpha)^2}{2}\left(\int_0^1 \varsigma(1-\varsigma)\,d\varsigma\right)^{1-\frac{1}{q}} \\ & \times\left(\int_0^1 \varsigma(1-\varsigma)\left|\varphi''(\varsigma\alpha+(1-\varsigma)\beta)\right|^q d\varsigma\right)^{\frac{1}{q}}, \end{aligned} \tag{61}$$

with $q \geq 1$.

Since $\int_0^1 \varsigma(1-\varsigma)\,d\varsigma = \frac{1}{6}$, it follows that:

$$\begin{aligned}\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta \varphi(z)\,dz\right| \leq & \frac{(\beta-\alpha)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}} \\ & \times\left(\int_0^1 \varsigma(1-\varsigma)\left|\varphi''(\varsigma\alpha+(1-\varsigma)\beta)\right|^q d\varsigma\right)^{\frac{1}{q}}. \end{aligned} \tag{62}$$

Utilizing Definition 4 for $s = 1$, it holds that:

$$\begin{aligned}
&\int_0^1 \varsigma(1-\varsigma)\,|\varphi''(\varsigma\alpha+(1-\varsigma)\beta)|^q\,d\varsigma\\
&\le \int_0^1 \varsigma(1-\varsigma)\,[\varsigma|\varphi''(\alpha)|^q+(1-\varsigma)\,|\varphi''(\beta)|^q]\;d\varsigma\\
&\le \int_0^1 \varsigma(1-\varsigma)\left[\frac{\sqrt{\varsigma}}{2\sqrt{1-\varsigma}}|\varphi''(\alpha)|^q+\frac{\sqrt{1-\varsigma}}{2\sqrt{\varsigma}}|\varphi''(\beta)|^q\right]d\varsigma\\
&=\frac{1}{2}\int_0^1 \varsigma(1-\varsigma)\frac{\sqrt{\varsigma}}{\sqrt{1-\varsigma}}|\varphi''(\alpha)|^q\,d\varsigma+\frac{1}{2}\int_0^1 \varsigma(1-\varsigma)\frac{\sqrt{1-\varsigma}}{\sqrt{\varsigma}}|\varphi''(\beta)|^q\,d\varsigma\\
&=\frac{1}{2}\int_0^1 \varsigma^{3/2}(1-\varsigma)^{1/2}|\varphi''(\alpha)|^q\,d\varsigma+\frac{1}{2}\int_0^1 \varsigma^{1/2}(1-\varsigma)^{3/2}|\varphi''(\beta)|^q\,d\varsigma\\
&=\frac{1}{2}\Bigg[\left(\int_0^1 \varsigma^{3/2}(1-\varsigma)^{1/2}\,d\varsigma\right)|\varphi''(\alpha)|^q\\
&\qquad+\left(\int_0^1 \varsigma^{1/2}(1-\varsigma)^{3/2}\,d\varsigma\right)|\varphi''(\beta)|^q\Bigg]. \qquad (63)
\end{aligned}$$

We note that both integrals in (63) are equal to the Euler beta function:

$$\begin{aligned}
\int_0^1 \varsigma^{3/2}(1-\varsigma)^{1/2}\,d\varsigma &= \int_0^1 \varsigma^{1/2}(1-\varsigma)^{3/2}\,d\varsigma\\
&= \mathrm{B}\left(\frac{5}{2},\frac{3}{2}\right)=\frac{\pi}{16}. \qquad (64)
\end{aligned}$$

Substituting (64) into (63), we find:

$$\begin{aligned}
\int_0^1 \varsigma(1-\varsigma)\,|\varphi''(\varsigma\alpha+(1-\varsigma)\beta)|^q\,d\varsigma &\le \frac{1}{2}\cdot\frac{\pi}{16}\left(|\varphi''(\alpha)|^q+|\varphi''(\beta)|^q\right)\\
&= \frac{\pi}{32}\left(|\varphi''(\alpha)|^q+|\varphi''(\beta)|^q\right). \qquad (65)
\end{aligned}$$

Plugging (65) back into (62), we get:

$$\begin{aligned}
\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta \varphi(z)\,dz\right| &\le \frac{(\beta-\alpha)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{\pi}{32}\right)^{\frac{1}{q}}\\
&\quad\times\left(|\varphi''(\alpha)|^q+|\varphi''(\beta)|^q\right)^{\frac{1}{q}}. \qquad (66)
\end{aligned}$$

Finally, the boundedness hypothesis of $|\varphi''|$ leads to:

$$\begin{aligned}
\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta \varphi(z)\,dz\right| &\le \frac{(\beta-\alpha)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{\pi}{32}\right)^{\frac{1}{q}}2^{\frac{1}{q}}M\\
&= \frac{(\beta-\alpha)^2}{4}\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{\pi}{8}\right)^{\frac{1}{q}}M. \qquad (67)
\end{aligned}$$

Which was to be proved. □

Remark 27. *Corollary 57 is equivalent to Theorem 2.4 of* [26].

Corollary 12. *The following estimate remains valid under the assumptions of Corollary 57:*

$$\left|\frac{\varphi(\alpha)+\varphi(\beta)}{2}-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\varphi(z)\,dz\right|\leq\frac{(\beta-\alpha)^2}{4}\mathrm{M}\left[\mathrm{B}\left(\frac{3}{2},q+\frac{1}{2}\right)\right]^{\frac{1}{q}}.$$

Here, $\mathrm{B}(\cdot,\cdot)$ *denotes the Euler beta function.*

Remark 28. *Corollary 59 is exactly Theorem 2.5 of* [26].

Theorem 9. *Let be a function* φ *defined over a closed interval* $D\subset\mathbb{R}$*, which it is twice differentiable in the interior* D^o*. Consider a weight function* $w:[0,1]\to\mathbb{R}$ *and* $\Lambda(\varsigma,\vartheta,\rho)=\Phi(\varsigma)\vartheta+\Psi(\varsigma)\rho$ *a weighted path with* $\vartheta,\rho\in[\alpha,\beta]\subset D^o$*. Suppose that* $|\varphi''|^q$ *satisfies the* (Λ,δ)*- convexity condition of second type over* $\left[\alpha,\frac{\beta}{m}\right]\subset D^o$*, then it's true that:*

$$\begin{aligned}|\mathbf{T}_1| \quad\leq\quad & (\mathcal{M}_1)^2\left[\frac{1}{p}\left(\int_0^1 w^p(\varsigma)\,d\varsigma\right)^{\frac{1}{p}}+\frac{1}{q}\Bigg(|\varphi''(\alpha)|^q\int_0^1\Phi\big(\delta^s(\varsigma)\big)\,d\varsigma\right.\\ &\left.+m\left|\varphi''\big(\tfrac{\kappa}{m}\big)\right|^q\int_0^1\big(\Psi(\delta(\varsigma))\big)^s\,d\varsigma\Bigg)^{\frac{1}{q}}\right]\\ &+(\mathcal{M}_2)^2\left[\frac{1}{p}\left(\int_0^1 w^p(\varsigma)\,d\varsigma\right)^{\frac{1}{p}}+\frac{1}{q}\Bigg(|\varphi''(\beta)|^q\int_0^1\Phi\big(\delta^s(\varsigma)\big)\,d\varsigma\right.\\ &\left.+m\left|\varphi''\big(\tfrac{\kappa}{m}\big)\right|^q\int_0^1\big(\Psi(\delta(\varsigma))\big)^s\,d\varsigma\Bigg)^{\frac{1}{q}}\right]+|\mathbf{E}_1|+|\mathbf{E}_2|.\end{aligned}$$

for $\kappa\in[\alpha,\beta]$ *and* $\mathbf{T}_1$ *defined in (10).*

Proof: We know that:

$$\begin{aligned}|\mathbf{T}_1| \quad\leq\quad & (\mathcal{M}_1)^2\int_0^1 w(\varsigma)\left|\varphi''\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\right|\,d\varsigma+|\mathbf{E}_1|\\ &+(\mathcal{M}_2)^2\int_0^1 w(\varsigma)\left|\varphi''\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\right|\,d\varsigma+|\mathbf{E}_2|.\end{aligned}\tag{68}$$

Employing Young's inequality in (69), we get:

$$\begin{aligned}|\mathbf{T}_1| \quad\leq\quad & (\mathcal{M}_1)^2\left[\frac{1}{p}\left(\int_0^1 w^p(\varsigma)\,d\varsigma\right)^{\frac{1}{p}}+\frac{1}{q}\left(\int_0^1\left|\varphi''\left(\Lambda_{\alpha,\kappa}(\varsigma)\right)\right|^q dt\right)^{\frac{1}{q}}\right]\\ &+(\mathcal{M}_2)^2\left[\frac{1}{p}\left(\int_0^1 w^p(\varsigma)\,d\varsigma\right)^{\frac{1}{p}}+\frac{1}{q}\left(\int_0^1\left|\varphi''\left(\Lambda_{\beta,\kappa}(\varsigma)\right)\right|^q dt\right)^{\frac{1}{q}}\right]+|\mathbf{E}_1|+|\mathbf{E}_2|,\end{aligned}\tag{69}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Utilizing the second-type (Λ,δ)- convexity property of $|\varphi''|^q$ and from (69), we obtain:

$$
\begin{aligned}
|\mathbf{T}_1| \quad \leq \quad & (\mathcal{M}_1)^2 \left[\frac{1}{p} \left(\int_0^1 w^p(\varsigma)\, d\varsigma \right)^{\frac{1}{p}} + \frac{1}{q} \Bigg(|\varphi''(\alpha)|^q \int_0^1 \Phi\big(\delta^s(\varsigma)\big)\, d\varsigma \right. \\
& \left. + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \left(\Psi(\delta(\varsigma))\right)^s d\varsigma \Bigg)^{\frac{1}{q}} \right] \\
& + (\mathcal{M}_2)^2 \left[\frac{1}{p} \left(\int_0^1 w^p(\varsigma)\, d\varsigma \right)^{\frac{1}{p}} + \frac{1}{q} \Bigg(|\varphi''(\beta)|^q \int_0^1 \Phi\big(\delta^s(\varsigma)\big)\, d\varsigma \right. \\
& \left. + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \left(\Psi(\delta(\varsigma))\right)^s d\varsigma \Bigg)^{\frac{1}{q}} \right] + |\mathbf{E}_1| + |\mathbf{E}_2|.
\end{aligned}
$$

As we wanted to demonstrate. □

Corollary 13. *Under the hypotheses of Theorem 61 but assume that $|\varphi''|^q$ is a modified (h, m)-convex function of the second type. Then*

$$
\begin{aligned}
|\mathbf{T}_2| \quad \leq \quad & \frac{|\alpha-\kappa|^2 + |\beta-\kappa|^2}{p\,(\lambda+1)} \left(\int_0^1 w^p(\varsigma)\, d\varsigma \right)^{\frac{1}{p}} + \frac{1}{q} \left[|\varphi''(\alpha)|^q \int_0^1 \delta^s \left(\frac{\lambda+\varsigma}{\lambda+1} \right) d\varsigma \right. \\
& \left. + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \left(1 - \delta\left(\frac{\lambda+\varsigma}{\lambda+1} \right) \right)^s d\varsigma \right]^{\frac{1}{q}} \\
& + \left(\frac{\beta-\kappa}{\lambda+1} \right)^2 \left\{ \frac{1}{q} \left[|\varphi''(\beta)|^q \int_0^1 \delta^s \left(\frac{\lambda+\varsigma}{\lambda+1} \right) d\varsigma \right. \right. \\
& \left. \left. + m \left|\varphi''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \left(1 - \delta\left(\frac{\lambda+\varsigma}{\lambda+1} \right) \right)^s d\varsigma \right]^{\frac{1}{q}} \right\},
\end{aligned}
$$

with $\mathbf{T}_2$ *introduced in* (11).

Proof: Invoking the expression proved earlier:

$$
\begin{aligned}
|\mathbf{T}_2| \quad \leq \quad & \left| \frac{\alpha-\kappa}{\lambda+1} \right|^2 \int_0^1 w(\varsigma) \left| \varphi'' \left(\frac{\lambda+\varsigma}{\lambda+1}\alpha + \frac{1-\varsigma}{\lambda+1}\kappa \right) \right| d\varsigma \\
& + \left| \frac{\beta-\kappa}{\lambda+1} \right|^2 \int_0^1 w(\varsigma) \left| \varphi'' \left(\frac{\lambda+\varsigma}{\lambda+1}\beta + \frac{1-\varsigma}{\lambda+1}\kappa \right) \right| d\varsigma.
\end{aligned} \tag{70}
$$

By means of Young's inequality applied to (70), one obtains:

$$
\begin{aligned}
|\mathbf{T}_2| \leq & \frac{|\alpha-\kappa|^2 + |\beta-\kappa|^2}{p\,(\lambda+1)} \left(\int_0^1 w^p(\varsigma)\, d\varsigma \right)^{\frac{1}{p}} \\
& + \frac{1}{q} \left[\left| \frac{\alpha-\kappa}{\lambda+1} \right|^2 \left(\int_0^1 \left| \varphi'' \left(\frac{\lambda+\varsigma}{\lambda+1}\alpha + \frac{1-\varsigma}{\lambda+1}\kappa \right) \right|^q d\varsigma \right)^{\frac{1}{q}} \right. \\
& \left. + \left| \frac{\beta-\kappa}{\lambda+1} \right|^2 \left(\int_0^1 \left| \varphi'' \left(\frac{\lambda+\varsigma}{\lambda+1}\beta + \frac{1-\varsigma}{\lambda+1}\kappa \right) \right|^q d\varsigma \right)^{\frac{1}{q}} \right],
\end{aligned} \tag{71}
$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Utilizing the (h, m)-convexity of $|\varphi''|^q$ in (71), the desired result follows. □

Remark 29. *Corollary 62 reproduces same statement that Theorem 2.20 of* [22].

Remark 30. *For $\varphi(\varsigma) = (f * g)(\varsigma)$, leads to an alternative formulation of convolution operators:*

$$
\begin{aligned}
|\mathbf{T}_3| \leq\ & \frac{|\alpha-\kappa|^2+|\beta-\kappa|^2}{p\,(\lambda+1)}\left(\int_0^1 w^p(\varsigma)\,d\varsigma\right)^{\frac{1}{p}} + \frac{1}{q}\left[|(f*g)''(\alpha)|^q \int_0^1 \delta^s\left(\frac{\lambda+\varsigma}{\lambda+1}\right)d\varsigma\right. \\
&\left. + m\left|(f*g)''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \left(1-\delta\left(\frac{\lambda+\varsigma}{\lambda+1}\right)\right)^s d\varsigma\right]^{\frac{1}{q}} \\
&+\left(\frac{\beta-\kappa}{\lambda+1}\right)^2 \left\{\frac{1}{q}\left[|(f*g)''(\beta)|^q \int_0^1 \delta^s\left(\frac{\lambda+\varsigma}{\lambda+1}\right)d\varsigma\right.\right. \\
&\left.\left. + m\left|(f*g)''\left(\tfrac{\kappa}{m}\right)\right|^q \int_0^1 \left(1-\delta\left(\frac{\lambda+\varsigma}{\lambda+1}\right)\right)^s d\varsigma\right]^{\frac{1}{q}}\right\},
\end{aligned}
$$

where $\mathbf{T}_3$ is given by (12).

3|Conclusions

In this paper, we established a broad family of Hermite–Hadamard type inequalities for twice differentiable functions within the framework of weighted paths and extended weighted fractional integral operators. The main contributions of this work can be summarized as follows.

A Fundamental Identity. The cornerstone of this study is the derivation of a new integral identity involving the second derivative of the underlying function. This identity provides a unified starting point for obtaining a variety of estimates and extends several classical identities previously used in the literature.

General Framework Based on Weighted Paths. By introducing the weighted path Λ together with the concept of (Λ, δ)-convexity, the obtained results are formulated in a highly flexible setting. This approach substantially enlarges the class of admissible convex functions and includes numerous existing notions of convexity as particular cases through suitable choices of Φ, Ψ, δ, m, and s.

Refined Hermite–Hadamard Inequalities. By combining the fundamental identity with Hölder's, Power Mean and Young's inequalities, several new estimates were established for twice differentiable functions whose second derivatives satisfy generalized convexity assumptions. These bounds provide quantitative refinements that extend many known Hermite–Hadamard type inequalities.

Recovery of Classical Results. An important feature of the proposed framework is its unifying character. Appropriate specifications of the weight function and the interpolation path recover several well-known inequalities involving the classical integral, weighted fractional integrals, Riemann–Liouville, Caputo–Fabrizio and Atangana–Baleanu fractional integral operators. This confirms the consistency and generality of the developed theory.

As natural extensions of this research, the study of these inequalities in the context of coordinate functions (convexity in the plane) is proposed, as well as the exploration of applications in information theory and convex optimization, where the control of integral bounds is critical for the convergence of algorithms.

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Author Contribution

L.F.A Gómez: Conceptualization, Formal analysis, Visualization, Investigation, Writing – original draft.
J.E. Nápoles Valdés: Validation, Funding acquisition, Methodology, Project administration, Resources, Supervision, Writing – review & editing.
All authors have read and agreed to the published version of the manuscript.

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Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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