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Solitonic Solutions for the Coupled Burgers Equation in Space-Time Using Composite Derivatives

Juan E. Nápoles V.[1,2]

[1] UNNE, FaCENA, Ave. Libertad 5450, Corrientes 3400, Argentina; jnapoles@exa.unne.edu.ar.
[2] UTN-FRRE, French 414, Resistencia, Chaco 3500, Argentina.

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Abstract

In this article, we investigate the coupled Burgers equation in spacetime governed by the local composite derivative CN. By defining a generalized traveling variable transformation, we reduce the coupled system of partial differential equations to integrable ordinary differential equations. Using a hyperbolic ansatz, we derive soliton solutions and evaluate them for three representative kernel classes: potential, exponential, and Mittag-Leffler. We also show that the conformable derivative formulation represents only a restricted monomial subset of our generalized integral framework. Numerical simulations and contour visualizations illustrate how the choice of kernel distorts the wavefront geometry and phase propagation in nonhomogeneous media.

Keywords: Coupled Burgers' equation, Composite CN-derivative, Exact soliton solutions, Integral traveling wave variable.

1|Introduction

Unlike other local operators proposed in recent literature, which are often criticized for simply rescaling the ordinary derivative using the chain rule, the composite derivative (CN-derivative) introduces a non-trivial deformation of the domain that formally extends the class of differentiable functions. By evaluating the local rate of change over an argument transformed by a kernel $N(x, \alpha)$, this operator allows us to analyze and differentiate functions that exhibit sharp points or vertical tangents in the classical sense, while preserving a manageable

Corresponding Author: jnapoles@exa.unne.edu.ar
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algebraic structure. This ability to re-metrize space and time makes the composite derivative a powerful analytical tool for modeling complex nonlinear phenomena in heterogeneous media and with non-uniform spatiotemporal response rates.

In [6] we define the composite derivative, as follows:

Definition 1. *Given a function $f : [0, +\infty) \to \mathbb{R}$. Then the CN-derivative of f of order α (the composite N derivative) is defined by*

$$f_{CN}^{\alpha}(a) = \lim_{x\to a} \frac{f(N(x,\alpha)) - f(N(a,\alpha))}{x-a}, \qquad \alpha \in (0,1].$$

If f is differentiable at N, and N is differentiable at a, then

$$f_{CN}^{\alpha}(a) = f'_N(a)N'(a,\alpha)$$

We must point out that, unlike other local operators, where the criticism focuses on the fact that they are a simple rescaling of the ordinary derivative, this composite derivative expands the class of differentiable functions.

1.1|Examples of Functions Not Classically Differentiable but Differentiable in the CN Sense

The fundamental property of the CN operator is that the function f does not need to be differentiable at the original point a, but only at the transformed point $N(a,\alpha)$. Furthermore, the appropriate choice of kernel $N(x,\alpha)$ can "smooth" out sharp corners, peaks, or classical non-differentiable behavior.

Below are two examples of functions that are not differentiable in the classical sense but are differentiable in the composite sense.

Example 1. *Consider the absolute value function $f(t) = |t|$ at the origin $t = 0$.*

Left-hand limit: $\lim_{t\to 0^-} \frac{|t|-0}{t} = -1$

Right-hand limit: $\lim_{t\to 0^+} \frac{|t|-0}{t} = 1$

Since the left-hand and right-hand limits do not coincide, the classical derivative $f'(0)$ does not exist (it has a corner).

Analysis with Composite Derivatives (CN). Let's choose a smooth shift kernel for the variable, for example:

$$N(t,\alpha) = t + (1-\alpha)$$

for a fixed parameter $\alpha \in (0,1)$.

Evaluating the kernel at $t = 0$:

$$N(0,\alpha) = 1 - \alpha > 0$$

Behavior of f at $N(0,\alpha)$: For any $\alpha \in (0,1)$, the value $N(0,\alpha)$ is strictly positive. Since $f(x) = |x| = x$ for all $x > 0$, the function f is completely differentiable in the neighborhood of the point $N(0,\alpha)$, with derivative $f'(N(0,\alpha)) = 1$.

Let's corroborate this by the definition:

$$f_{CN}^{\alpha}(0) = \lim_{t\to 0} \frac{|N(t,\alpha)| - |N(0,\alpha)|}{t-0} = \lim_{t\to 0} \frac{|t+1-\alpha| - (1-\alpha)}{t}$$

Since $1-\alpha > 0$, for t close enough to 0, we have that $t + 1 - \alpha > 0$:

$$f_{CN}^{\alpha}(0) = \lim_{t\to 0} \frac{(t+1-\alpha) - (1-\alpha)}{t} = \lim_{t\to 0} \frac{t}{t} = 1$$

If we calculate using the kernel formula, given that $N'(t,\alpha) = \frac{d}{dt}(t+1-\alpha) = 1$:

$$f_{CN}^{\alpha}(0) = f'\left(N(0,\alpha)\right) \cdot N'(0,\alpha) = f'(1-\alpha) \cdot 1 = 1 \cdot 1 = 1$$

The function $f(t) = |t|$ is not differentiable at $t = 0$ in the classical sense, but it is differentiable at $t = 0$ in the CN sense with $f_{CN}^{\alpha}(0) = 1$.

Example 2. *Consider the function $f(t) = t^{1/3}$ at $t = 0$.*

$f'(t) = \frac{1}{3}t^{-2/3}$ *Evaluating at the origin,* $f'(0) = \lim_{t\to 0} \frac{t^{1/3}-0}{t} = \lim_{t\to 0} \frac{1}{t^{2/3}} = +\infty$

The function has a vertical tangent at $t = 0$, so the classical derivative $f'(0)$ does not exist in $\mathbb{R}$.

We choose a kernel of the shifted quadratic power type:

$$N(t,\alpha) = (t+1)^{3/\alpha} - 1 \quad \text{or simply a potential kernel} \quad N(t,\alpha) = (t+\alpha)^3$$

For greater algebraic simplicity, let's take $N(t,\alpha) = (t+\alpha)^3$ with $\alpha > 0$.

Evaluating the kernel at $t = 0$:

$$N(0,\alpha) = \alpha^3 > 0$$

Ordinary derivative of the kernel:

$$N'(t,\alpha) = \frac{\partial}{\partial t}(t+\alpha)^3 = 3(t+\alpha)^2 \implies N'(0,\alpha) = 3\alpha^2$$

Evaluating f at the kernel:

$$f(N(t,\alpha)) = \left((t+\alpha)^3\right)^{1/3} = t+\alpha$$

Note that the composition $f(N(t,\alpha))$ transforms the function with a vertical tangent into a smooth linear polynomial $t+\alpha$.

Applying the definition by limit:

$$f_{CN}^{\alpha}(0) = \lim_{t\to 0} \frac{f(N(t,\alpha)) - f(N(0,\alpha))}{t-0} = \lim_{t\to 0} \frac{(t+\alpha)-\alpha}{t} = \lim_{t\to 0} \frac{t}{t} = 1$$

Verification by core product rule:

$$f_N'(0) = f'\left(N(0,\alpha)\right) = f'(\alpha^3) = \frac{1}{3}(\alpha^3)^{-2/3} = \frac{1}{3\alpha^2}$$

Multiplying by $N'(0,\alpha) = 3\alpha^2$:

$$f_{CN}^{\alpha}(0) = f_N'(0) \cdot N'(0,\alpha) = \left(\frac{1}{3\alpha^2}\right) \cdot \left(3\alpha^2\right) = 1$$

The function $f(t) = t^{1/3}$ does not have a finite derivative at $t = 0$ in the classical sense, but its composite derivative at $t = 0$ exists and is a finite real value ($f_{CN}^{\alpha}(0) = 1$).

The Burgers equation is one of the fundamental mathematical models in the physics of nonlinear phenomena, as it represents the simplest scalar formulation that balances nonlinear convection with diffusion or dissipation effects. Historically considered a one-dimensional simplification of the Navier-Stokes equations, its importance in current research transcends fluid mechanics, extending to hydrodynamics, nonlinear acoustics, traffic dynamics, cosmology (in the formation of large-scale structures in the universe), and shock wave propagation. In the context of contemporary science, the analysis of the Burgers equation using generalized differential operators represents a field of intense research, as it allows for the modeling of soliton and wavefront propagation in non-homogeneous, complex, or microstructured media, offering a key theoretical window for understanding realistic transport processes in systems far from equilibrium.

Initially, the basic reference to the Burger equation was introduced by J. M. Burger in 1974 ([1]). Hopf and Cole analyzed the Burger equation in the context of gas dynamics (see [3, 7]). Subsequently, Lian Yang et al. (in [9]) studied the small-amplitude, long-wavelength limit of the gas dynamics system for the Burger equation. Under the effect of gravity, the coupled Burger equation (1 + 1) (see [4]) is a simple model of the evolution and sedimentation of volumetric deposition at scales of two types of elements in fluid suspensions or colloids.

Let $N_1(t,\alpha)$ and $N_2(s,\beta)$ be the kernels associated with the time variables t and space variables s, respectively. Consider the coupled nonlinear partial system:

$$\begin{cases} w^{\alpha}_{CN}(t) - w^{2\beta}_{CN}(s) + 2w(s,t)\, w^{\beta}_{CN}(s) + k\,[w(s,t)z(s,t)]^{\beta}_{CN}(s) = 0 \\ z^{\alpha}_{CN}(t) - z^{2\beta}_{CN}(s) + 2z(s,t)\, z^{\beta}_{CN}(s) + l\,[w(s,t)z(s,t)]^{\beta}_{CN}(s) = 0 \end{cases} \tag{1}$$

where the second-order operator is equivalent to the iterated application $w^{2\beta}_{CN}(s) = \left(w^{\beta}_{CN}(s)\right)^{\beta}_{CN}(s)$, $s > 0$, $t > 0$, $0 < \alpha, \beta \geq 1$. In the case of N is the identity we have the conventionally coupled Burger's equation.

In [12], the coupled system of fractional Burgers equations in space-time defined in the sense of the conformable derivative was investigated:

$$\frac{\partial^{\alpha} w}{\partial t^{\alpha}} - \frac{\partial^{2\beta} w}{\partial s^{2\beta}} + 2w\frac{\partial^{\beta} w}{\partial s^{\beta}} + k\frac{\partial^{\beta}(wz)}{\partial s^{\beta}} = 0$$

$$\frac{\partial^{\alpha} z}{\partial t^{\alpha}} - \frac{\partial^{2\beta} z}{\partial s^{2\beta}} + 2z\frac{\partial^{\beta} z}{\partial s^{\beta}} + l\frac{\partial^{\beta}(wz)}{\partial s^{\beta}} = 0$$

where $t > 0$, $s > 0$, $0 < \alpha, \beta \leq 1$, and the derivatives $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ and $\frac{\partial^{\beta}}{\partial s^{\beta}}$ are the local operators of the introduced conformable derivative in [8].

The main objective of this work is to extend the framework of coupled nonlinear systems by studying the spacetime-coupled Burgers equation governed by the composite derivative CN. Employing a generalized traveling variable transformation defined by integrals over the upper bound, we reduce the system of partial differential equations to a system of integrable ordinary differential equations. Furthermore, using an ansatz of hyperbolic functions, we obtain exact soliton solutions and evaluate them for three distinct kernel classes: potential, exponential, and Mittag-Leffler. This approach demonstrates how the choice of kernel distorts the waveform profile and phase geometry, providing a more general framework than that of conventional and conformable local operators.

2|Results

Theorem 1. *Under the composite traveling variable transformation*

$$\xi(s,t) = m\int_0^s \frac{d\tau}{N_2(\tau,\beta)} - n\int_0^t \frac{d\tau}{N_1(\tau,\alpha)}$$

with $w(s,t) = W(\xi)$ and $z(s,t) = Z(\xi)$ (where $m, n \in \mathbb{R} \setminus \{0\}$), the nonlinear system 1 reduces to the following system of integrated ordinary differential equations (fixing the constant of integration to zero):

$$-nW(\xi) - m^2 W'(\xi) + mW^2(\xi) + kmW(\xi)Z(\xi) = 0$$
$$-nZ(\xi) - m^2 Z'(\xi) + mZ^2(\xi) + lmW(\xi)Z(\xi) = 0$$

Proof: Calculating the derivative of the traveling variable $\xi(s,t)$ we have:

$$\frac{\partial \xi}{\partial s} = \frac{\partial}{\partial s}\left(m\int_0^s \frac{d\tau}{N_2(\tau,\beta)}\right) = \frac{m}{N_2(s,\beta)}$$

$$\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial t}\left(-n\int_0^t \frac{d\tau}{N_1(\tau,\alpha)}\right) = \frac{-n}{N_1(t,\alpha)}$$

Applying Definition 1 to $w(s,t) = W(\xi(s,t))$ in the variable t:

$$w^{\alpha}_{CN}(t) = \left[\frac{\partial w}{\partial t}\right]_{t\to N_1(t,\alpha)} \cdot N_1(t,\alpha)$$

Using the chain rule for the ordinary partial derivative:

$$\frac{\partial w}{\partial t} = W'(\xi)\frac{\partial \xi}{\partial t} = W'(\xi)\left(\frac{-n}{N_1(t,\alpha)}\right)$$

Substituting into the definition of $w^{\alpha}_{CN}(t)$:

$$w^{\alpha}_{CN}(t) = \left(W'(\xi)\frac{-n}{N_1(t,\alpha)}\right) \cdot N_1(t,\alpha) = -nW'(\xi)$$

Identically for the spatial coordinate s:

$$w^{\beta}_{CN}(s) = \left[\frac{\partial w}{\partial s}\right]_{s\to N_2(s,\beta)} \cdot N_2(s,\beta) = \left(W'(\xi)\frac{m}{N_2'(s,\beta)}\right) \cdot N_2(s,\beta) = mW'(\xi)$$

For the double composition $\left(w^{\beta}_{CN}\right)^{\beta}_{CN}(s)$, let us denote the intermediate function $g(s) = w^{\beta}_{CN}(s) = mW'(\xi(s,t))$. Applying the composite operator on $g(s)$:

$$\left(w^{\beta}_{CN}\right)^{\beta}_{CN}(s) = g^{\beta}_{CN}(s) = \left[\frac{\partial g}{\partial s}\right]_{s\to N_2(s,\beta)} \cdot N_2(s,\beta)$$

Let's calculate the ordinary derivative of $g(s)$:

$$\frac{\partial g}{\partial s} = \frac{\partial}{\partial s}\left(mW'(\xi)\right) = mW''(\xi)\frac{\partial \xi}{\partial s} = mW''(\xi)\left(\frac{m}{N_2(s,\beta)}\right)$$

Substituting in the composition double:

$$\left(w^{\beta}_{CN}\right)^{\beta}_{CN}(s) = \left(m^2 W''(\xi)\frac{1}{N_2'(s,\beta)}\right) \cdot N_2(s,\beta) = m^2 W''(\xi)$$

Similarly, for the nonlinear term of the product and the components of $z(s,t) = Z(\xi)$:

$$[w(s,t)z(s,t)]^{\beta}_{CN}(s) = m(WZ)'(\xi)$$

$$z^{\alpha}_{CN}(t) = -nZ'(\xi), \quad z^{\beta}_{CN}(s) = mZ'(\xi), \quad \left(z^{\beta}_{CN}\right)^{\beta}_{CN}(s) = m^2 Z''(\xi)$$

Substituting the simplified expressions in the first ODE:

$$-nW'(\xi) - m^2 W''(\xi) + 2mW(\xi)W'(\xi) + km(WZ)'(\xi) = 0$$

Since $2WW' = (W^2)'$, we group under the ordinary derivative with respect to ξ:

$$\frac{d}{d\xi}\left(-nW(\xi) - m^2 W'(\xi) + mW^2(\xi) + kmW(\xi)Z(\xi)\right) = 0$$

Integrating with respect to ξ and setting the integration constant as 0:

$$-nW - m^2 W' + mW^2 + kmWZ = 0 \tag{2}$$

Performing the exact same procedure for the second PDE:

$$\frac{d}{d\xi}\left(-nZ(\xi) - m^2 Z'(\xi) + mZ^2(\xi) + lmW(\xi)Z(\xi)\right) = 0$$

Integrating with a constant of zero:

$$-nZ - m^2 Z' + mZ^2 + lmWZ = 0 \tag{3}$$

This completes the proof of the Theorem. □

Remark 1. *It is worth emphasizing that the conformable derivative used in [12] does not constitute an independent or more general operator, but a particular subcase of the composite derivative CN. Indeed, by selecting the specific potential kernel*

$$N_C(x,\alpha) = \frac{x^{\alpha}}{\alpha}, \quad \alpha \in (0,1],$$

whose derivative is $N'_C(x,\alpha) = x^{\alpha-1} = \frac{1}{x^{1-\alpha}}$, the composite derivative reduces directly to

$$f^{\alpha}_{CN}(x) = f'\left(\frac{x^{\alpha}}{\alpha}\right) \cdot \frac{1}{x^{1-\alpha}} \Longleftrightarrow D^{\alpha} f(x) = x^{1-\alpha}\frac{df(x)}{dx}.$$

Therefore, under the choice of spatial and temporal kernels $N_2(s,\beta) = \frac{s^{\beta}}{\beta}$ and $N_1(t,\alpha) = \frac{t^{\alpha}}{\alpha}$, our generalized traveling variable

$$\xi(s,t) = m\int_0^s \frac{d\tau}{N_2(\tau,\beta)} - n\int_0^t \frac{d\tau}{N_1(\tau,\alpha)}$$

reduces exactly to the conformable traveling variable $\xi_{conf}(s,t) = m\left(\frac{\beta}{1-\beta}s^{1-\beta}\right) - n\left(\frac{\alpha}{1-\alpha}t^{1-\alpha}\right)$, where the scaling factors are absorbed within the wave speed parameters m and n. This analytical reduction demonstrates rigorously that the results obtained in said work, represent a particular case of our composite framework CN, which additionally accommodates exponential, Mittag-Leffler behaviors and non-differentiable functions in the classical sense.

Theorem 2. *(Generalized Exact Soliton Solutions)*

Under the hyperbolic transformation given by $\frac{d\gamma}{d\xi} = \sinh(\xi)$ *(with* $\sinh(\gamma) = -csch(\xi)$ *and* $\cosh(\gamma) = -coth(\xi)$*), the generalized soliton solutions for* $kl \neq 1$ *are given by:*

$$w(s,t) = \frac{m(1-k)}{2(kl-1)}\left[1 + \coth\left(\xi(s,t)\right) - csch\left(\xi(s,t)\right)\right]$$

$$z(s,t) = \frac{m(1-l)}{2(kl-1)}\left[1 + \coth\left(\xi(s,t)\right) - csch\left(\xi(s,t)\right)\right]$$

where the exact traveling variable in the composite sense is defined as before.

Proof: In the reduced ODE $-nW - m^2W' + mW^2 + kmWZ = 0$, we balance the term with the highest derivative W' with the nonlinear term W^2:

$$\text{Degree}(W') = N+1, \quad \text{Degree}(W^2) = 2N \implies N+1 = 2N \implies N = 1$$

We formulate the hyperbolic polynomial ansatz of degree 1:

$$W(\xi) = A_0 + A_1\cosh(\gamma) + B_1\sinh(\gamma)$$

$$Z(\xi) = C_0 + C_1\cosh(\gamma) + D_1\sinh(\gamma)$$

Using $\frac{d\gamma}{d\xi} = \sinh(\xi)$ and the relation $\sinh(\xi) = -\text{csch}(\gamma)$:

$$W'(\xi) = \frac{dW}{d\gamma}\frac{d\gamma}{d\xi} = (A_1\sinh(\gamma) + B_1\cosh(\gamma))\sinh(\xi) = -A_1 - B_1\coth(\gamma)$$

Substituting $W(\xi)$, $Z(\xi)$ and $W'(\xi)$ into the reduced ODEs of Theorem 1 and setting each coefficient of the hyperbolic basis $\{1, \cosh(\gamma), \sinh(\gamma), \dots\}$ equal to zero, we solve the nonlinear algebraic system:

$$A_0 = \frac{m(1-k)}{2(kl-1)}, \quad A_1 = -\frac{m(1-k)}{2(kl-1)}, \quad B_1 = \frac{m(1-k)}{2(kl-1)}$$

$$C_0 = \frac{m(1-l)}{2(kl-1)}, \quad C_1 = -\frac{m(1-l)}{2(kl-1)}, \quad D_1 = \frac{m(1-l)}{2(kl-1)}$$

Substituting the coefficients obtained in $W(\xi)$:

$$W(\xi) = \frac{m(1-k)}{2(kl-1)}\left[1 - \cosh(\gamma) + \sinh(\gamma)\right]$$

Replacing $\cosh(\gamma) = -\coth(\xi)$ and $\sinh(\gamma) = -\text{csch}(\xi)$:

$$W(\xi) = \frac{m(1-k)}{2(kl-1)}\left[1 - (-\coth(\xi)) + (-\text{csch}(\xi))\right] = \frac{m(1-k)}{2(kl-1)}\left[1 + \coth(\xi) - \text{csch}(\xi)\right]$$

Analogously for $Z(\xi)$:

$$Z(\xi) = \frac{m(1-l)}{2(kl-1)}\left[1 + \coth(\xi) - \text{csch}(\xi)\right]$$

Substituting the defined traveling variable, and given that the solutions in continuous space are given by $w(s,t) = W(\xi(s,t))$ and $z(s,t) = Z(\xi(s,t))$, we evaluate ξ in the hyperbolic expressions:

For the first component $w(s,t)$:

$$w(s,t) = \frac{m(1-k)}{2(kl-1)}\left[1 + \coth\left(m\int_0^s \frac{d\tau}{N_2'(\tau,\beta)} - n\int_0^t \frac{d\tau}{N_1'(\tau,\alpha)}\right) - \operatorname{csch}\left(m\int_0^s \frac{d\tau}{N_2'(\tau,\beta)} - n\int_0^t \frac{d\tau}{N_1'(\tau,\alpha)}\right)\right]$$

For the second component $z(s,t)$:

$$z(s,t) = \frac{m(1-l)}{2(kl-1)}\left[1 + \coth\left(m\int_0^s \frac{d\tau}{N_2'(\tau,\beta)} - n\int_0^t \frac{d\tau}{N_1'(\tau,\alpha)}\right) - \operatorname{csch}\left(m\int_0^s \frac{d\tau}{N_2'(\tau,\beta)} - n\int_0^t \frac{d\tau}{N_1'(\tau,\alpha)}\right)\right]$$

□

Remark 2. *To ensure the rigor that these solutions satisfy the original PDEs, let's check the cancellation of the kernels when deriving $w(s,t)$:*

Derivative with respect to t:

$$\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial t}\left(-n\int_0^t \frac{d\tau}{N_1'(\tau,\alpha)}\right) = -n\frac{1}{N_1'(t,\alpha)}$$

Applying the composite derivative:

$$w_{CN}^{\alpha}(t) = \left[\frac{\partial w}{\partial t}\right]_{t\to N_1(t,\alpha)} \cdot N_1'(t,\alpha) = \left(W'(\xi)\cdot \frac{-n}{N_1'(t,\alpha)}\right) N_1'(t,\alpha) = -nW'(\xi)$$

Derivative with respect to s:

$$\frac{\partial \xi}{\partial s} = \frac{\partial}{\partial s}\left(m\int_0^s \frac{d\tau}{N_2'(\tau,\beta)}\right) = m\frac{1}{N_2'(s,\beta)}$$

Applying the composite derivative and its double composition:

$$w_{CN}^{\beta}(s) = \left(W'(\xi)\cdot \frac{m}{N_2'(s,\beta)}\right) N_2'(s,\beta) = mW'(\xi)$$

$$\left(w_{CN}^{\beta}\right)_{CN}^{\beta}(s) = \left(mW''(\xi)\cdot \frac{m}{N_2'(s,\beta)}\right) N_2'(s,\beta) = m^2 W''(\xi)$$

Substituting these derivatives into the original coupled equations, the differential identity $0=0$ is trivially satisfied for all (s,t) in the domain of definition of the integrals.

3|Numerical Examples

To simplify the final numerical expressions and keep the focus on the action of the nuclei, we fix the following algebraic and system constants:

Burgers system parameters: $k=2$, $l=3$. Note that $kl-1 = (2)(3)-1 = 5 \neq 0$.

Amplitude coefficients:

$$\frac{m(1-k)}{2(kl-1)} = \frac{m(1-2)}{2(5)} = -\frac{m}{10}$$

$$\frac{m(1-l)}{2(kl-1)} = \frac{m(1-3)}{2(5)} = -\frac{2m}{10} = -\frac{m}{5}$$

Wave velocities/weights: $m = 2$, $n = 1$. Derivative orders $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$.

With these numerical parameters fixed, the solutions reduce to:

$$w(s,t) = -\frac{1}{5}\left[1 + \coth\left(\xi(s,t)\right) - \operatorname{csch}\left(\xi(s,t)\right)\right]$$

$$z(s,t) = -\frac{2}{5}\left[1 + \coth\left(\xi(s,t)\right) - \operatorname{csch}\left(\xi(s,t)\right)\right]$$

where the traveling variable takes the explicit form:

$$\xi(s,t) = 2\int_0^s \frac{d\tau}{N_2(\tau,\frac{1}{2})} - n\int_0^t \frac{d\tau}{N_1(\tau,\frac{1}{2})}$$

Let us consider three particular kernels to illustrate Theorem 3.

Example 3. Potential Kernel

$$N_1(t,1/2) = 2\sqrt{t}, \quad N_2(s,1/2) = 2\sqrt{s}$$

Space integral:

$$\int_0^s \frac{d\tau}{2\sqrt{\tau}} = \sqrt{s}$$

Time integral:

$$\int_0^t \frac{d\tau}{2\sqrt{\tau}} = \sqrt{t}$$

Traveling variable and Solutions

$$\xi_{Potential}(s,t) = 2\sqrt{s} - \sqrt{t}$$

$$w_1(s,t) = -\frac{1}{5}\left[1 + \coth\left(2\sqrt{s} - \sqrt{t}\right) - csch\left(2\sqrt{s} - \sqrt{t}\right)\right]$$

$$z_1(s,t) = -\frac{2}{5}\left[1 + \coth\left(2\sqrt{s} - \sqrt{t}\right) - csch\left(2\sqrt{s} - \sqrt{t}\right)\right]$$

Example 4. Exponential Kernel

$$N_1(t,1/2) = e^{t/2}, \quad N_2(s,1/2) = e^{s/2}$$

Space integral:

$$\int_0^s \frac{d\tau}{e^{\tau/2}} = \int_0^s e^{-\tau/2}d\tau = 2\left(1 - e^{-s/2}\right)$$

Time integral:

$$\int_0^t \frac{d\tau}{e^{\tau/2}} = 2\left(1 - e^{-t/2}\right)$$

Traveling variable and Solutions

$$\xi_{Exponential}(s,t) = 2\left[2(1 - e^{-s/2})\right] - 1\left[2(1 - e^{-t/2})\right] = 2 - 4e^{-s/2} + 2e^{-t/2}$$

$$w_2(s,t) = -\frac{1}{5}\left[1 + \coth\left(2 - 4e^{-s/2} + 2e^{-t/2}\right) - csch\left(2 - 4e^{-s/2} + 2e^{-t/2}\right)\right]$$

$$z_2(s,t) = -\frac{2}{5}\left[1 + \coth\left(2 - 4e^{-s/2} + 2e^{-t/2}\right) - csch\left(2 - 4e^{-s/2} + 2e^{-t/2}\right)\right]$$

Example 5. Mittag-Leffler Type Kernel (Regularized Sum)

Let's consider the linearized kernel associated with Mittag-Leffler:

$$N_1(t,1/2) = 1 + \sqrt{t}, \quad N_2(s,1/2) = 1 + \sqrt{s}$$

Space integral:

$$\int_0^s \frac{d\tau}{1 + \sqrt{\tau}}$$

Setting $u = \sqrt{\tau} \implies d\tau = 2u\,du$*:*

$$\int_0^{\sqrt{s}} \frac{2u\,du}{1+u} = 2\int_0^{\sqrt{s}} \left(1 - \frac{1}{1+u}\right) du = 2\sqrt{s} - 2\ln(1+\sqrt{s})$$

Time integral:

$$\int_0^t \frac{d\tau}{1 + \sqrt{\tau}} = 2\sqrt{t} - 2\ln(1+\sqrt{t})$$

Traveling variable and Solutions

$$\xi_{ML}(s,t) = 2\left[2\sqrt{s} - 2\ln(1+\sqrt{s})\right] - 1\left[2\sqrt{t} - 2\ln(1+\sqrt{t})\right]$$

$$\xi_{ML}(s,t) = 4\sqrt{s} - 2\sqrt{t} - 4\ln(1+\sqrt{s}) + 2\ln(1+\sqrt{t})$$

$$w_3(s,t) = -\frac{1}{5}\left[1 + \coth\left(\xi_{ML}(s,t)\right) - csch\left(\xi_{ML}(s,t)\right)\right]$$

$$z_3(s,t) = -\frac{2}{5}\left[1 + \coth\left(\xi_{ML}(s,t)\right) - csch\left(\xi_{ML}(s,t)\right)\right]$$

A heat map (or contour plot) for $w_1(s,t)$ visually represents how the amplitude of the soliton wave $w(s,t)$ varies simultaneously in space (s) and time (t) under the influence of a given nucleus. In other words, the heat map represents the spatiotemporal evolution of the soliton solution $w(s,t)$ for the composite-coupled Burgers equation. The color at each point (s,t) denotes the local value of the wave profile. Unlike conventional traveling waves that move at a constant speed along straight lines in the (s,t) plane, the colored bands in $W_{\text{potential}}$ exhibit a parabolic curvature (with potential kernel). This is because the composite traveling variable $\xi = 2\sqrt{s} - \sqrt{t}$ imposes a nonlinear relationship between position s and time t.

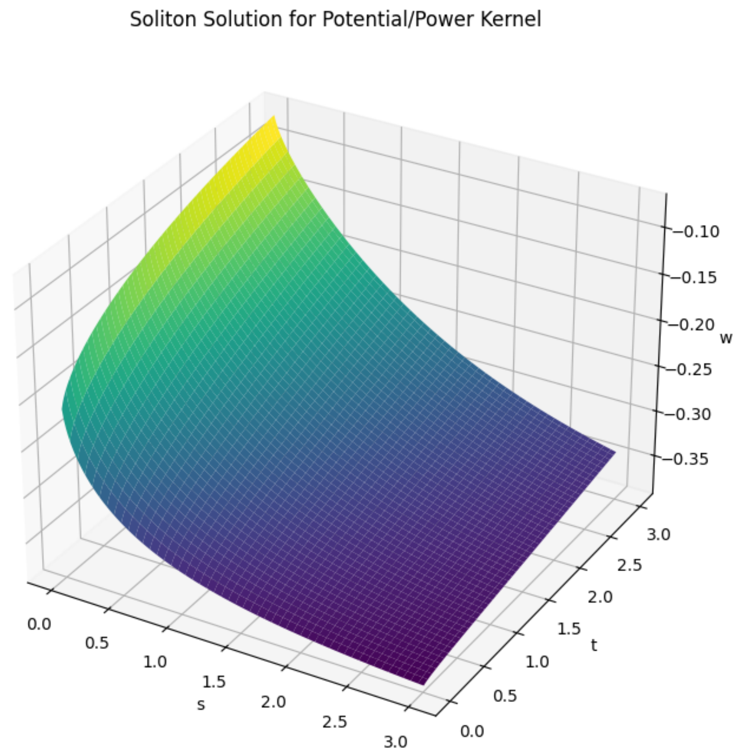
Soliton Solution for Potential/Power Kernel
−0.10
−0.15
−0.20
−0.25
−0.30
−0.35
w
0.0
0.5
1.0
1.5
2.0
2.5
3.0
s
t

Fig. 1.

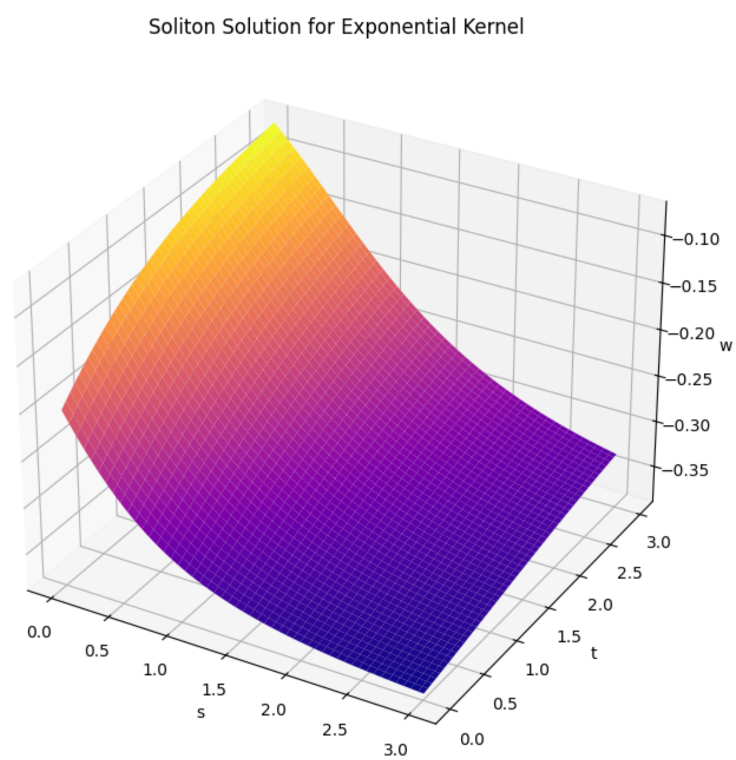
Soliton Solution for Exponential Kernel
−0.10
−0.15
−0.20
−0.25
−0.30
−0.35
w
0.0
0.5
1.0
1.5
2.0
2.5
3.0
s
t

Fig. 2.

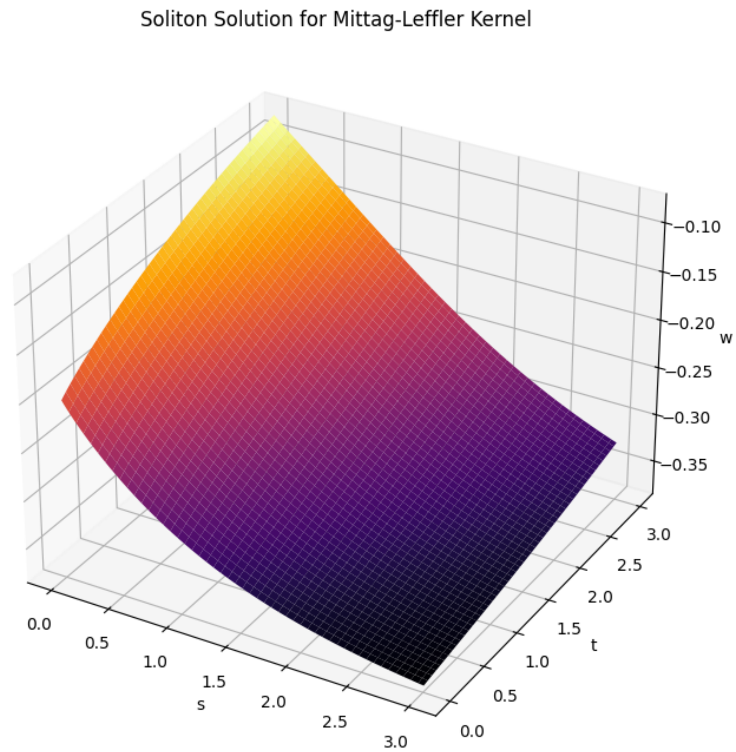
Soliton Solution for Mittag-Leffler Kernel
w
−0.10
−0.15
−0.20
−0.25
−0.30
−0.35
s
0.0
0.5
1.0
1.5
2.0
2.5
3.0
t
0.0
0.5
1.0
1.5
2.0
2.5
3.0

Fig. 3.

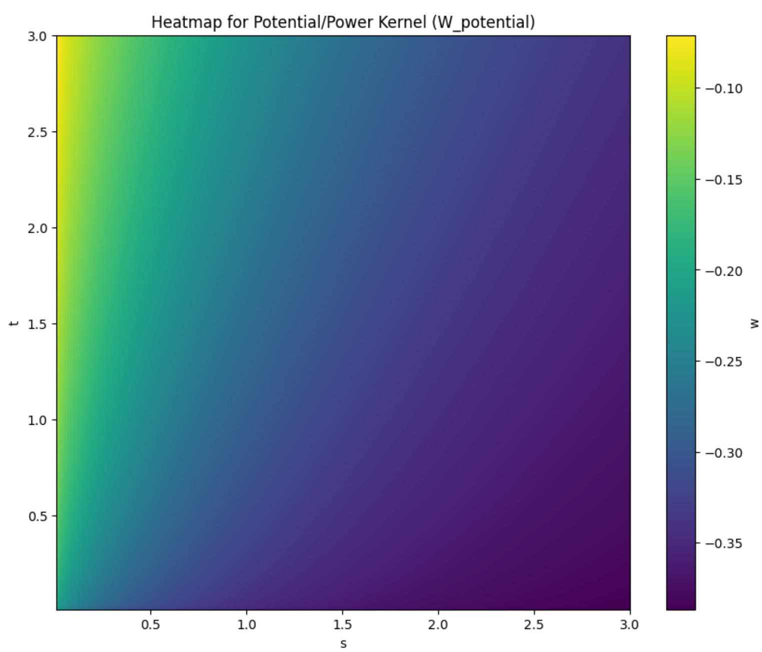
Heatmap for Potential/Power Kernel (W_potential)
t
0.5
1.0
1.5
2.0
2.5
3.0
s
0.5
1.0
1.5
2.0
2.5
3.0
w
−0.10
−0.15
−0.20
−0.25
−0.30
−0.35

Fig. 4.

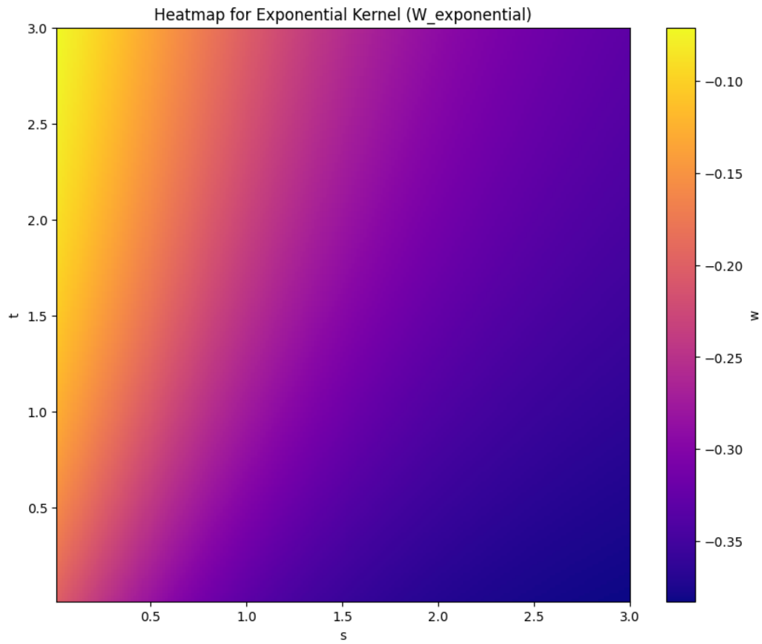

Fig. 5.

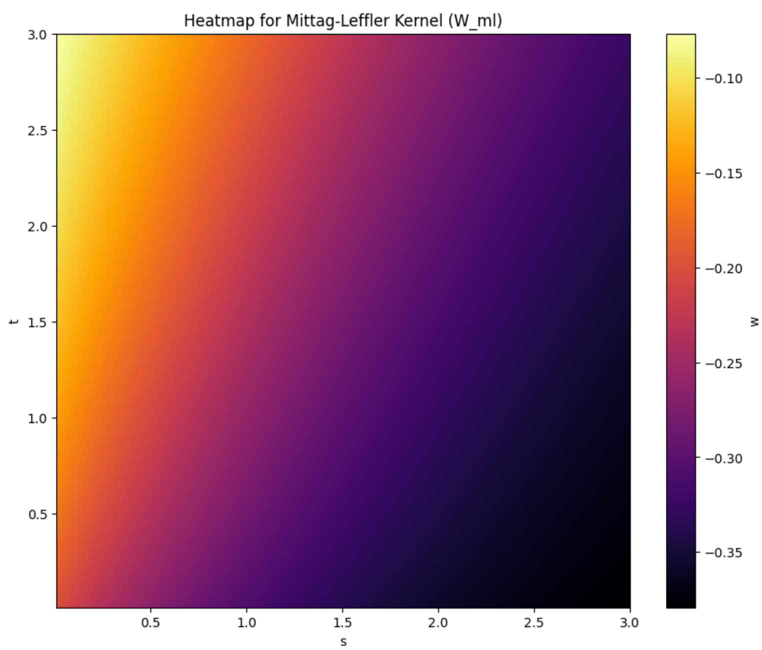

Fig. 6.

Remark 3. *The physical interpretation of the three heat maps for the generalized Burgers equation using the composite derivative focuses on how the mathematical nature of the kernel modifies the behavior of the medium, the effective spatiotemporal scaling, and the soliton's dynamics.*

The Burgers equation models nonlinear transport phenomena with dissipation (such as hydrodynamics, vehicular flow, dispersion in porous media, or the propagation of nonlinear acoustic waves). The composite operator alters the effective spacetime metric of the system, modifying the velocity and attenuation of the soliton wave $w(s,t)$*.*

1. Potential Nucleus / Power ($W_{potential}$)

Traveling Variable Equation $\xi = 2\sqrt{s} - \sqrt{t}$

Fractal Diffusion or Heterogeneous Media: The power-type dependence ($\sqrt{s}, \sqrt{t}$) models a medium with fractal geometry or scaled roughness, where the medium's resistance to flow varies with position.

Space-Time Curvature Relationship: Isoclines (wavefronts of constant value w) follow parabolas in the (s, t) plane. This implies that to maintain the same amplitude or phase of the wave as time t elapses, the required distance increases nonlinearly ($s \propto t$).

Progressive Deceleration: The wave experiences accelerated attenuation/dispersion in the initial stages ($t < 1$), stabilizing its slope as time progresses.

2. Exponential Kernel ($W_{exponential}$)

Equation of the traveling variable $\xi = 2 - 4e^{-s/2} + 2e^{-t/2}$

Deformed Dissipative Systems (Saturating Dynamic Rate): The exponential kernel represents a medium where the effective local response rate saturates exponentially over time/space, leading to a smooth transition toward a uniform propagation velocity.

Asymptotic Behavior and Saturation: Unlike the power case, the term $e^{-x/2}$ decays rapidly towards zero for large values of s and t. This means that the effective wave propagation speed saturates.

Transition Limit: For $s, t \leq 1$, the exponential terms vanish, forcing the traveling variable ξ to tend towards a constant value ($\xi \leq 2$). Physically, the wave reaches a stationary or equilibrium configuration in the dissipative medium, where nonlinearity and dissipation are completely balanced.

3. Mittag-Leffler Kernel (W_{ml})

Traveling Variable Equation $\xi = 4\sqrt{s} - 2\sqrt{t} - 4\ln(1+\sqrt{s}) + 2\ln(1+\sqrt{t})$

Non-linear Scale Invariance and Logarithmic Relaxation: The Mittag-Leffler kernel acts as a local scale-invariant modifier that interpolates smoothly between a power-law regime at small scales and a logarithmic scaling at large scales, modeling complex media with structural hierarchy (such as porous media or dense polymer chains) without memory integrals.

Aging Effect: The presence of the logarithmic term $\ln(1+\sqrt{x})$ acts as a "dissipative delay brake." The graph shows a much smoother and more extended color gradation than in the other two cases.

Transport in Complex Media: Describes the propagation of the soliton in materials with complex microstructure (such as gels, dense polymer fluids, or saturated porous reservoirs), where particles trapped in the microstructure cause a prolonged memory phenomenon on the amplitude of the wave $w(s, t)$.

4|Conclusion

In this work, we studied the coupled Burgers equation in spacetime governed by the composite derivative CN, establishing traveling-wave soliton solutions through an integral transformation as a function of the upper limit.

We demonstrate that the formulation using the classical conformable derivative represents a special case of our framework, with $N(x, \alpha) = x^{\alpha}/\alpha$, proving that the CN operator expands the class of admissible differentiable functions without resorting to non-local memory integrals.

Using numerical examples, we show that the choice of kernel fundamentally deforms the phase geometry (ξ = constant): the potential kernel induces nonlinear parabolic wavefronts, the exponential kernel forces asymptotic saturation towards uniform propagation, and the Mittag-Leffler kernel provides a smooth multiscale transition.

The unified solution framework offers a versatile tool for modeling the dynamics of nonlinear dissipative waves in complex, non-homogeneous, and microstructured media.

Furthermore, these results open new horizons using other local operators such as those defined in [2, 5, 10, 11].

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Author Contribution

All aspects of the research and manuscript preparation were carried out by the author. The author has read and approved the final version of the manuscript.

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Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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