

Existence of A Unique Weak Solution to the Second Boundary Value Problem for A Fully Nonlinear Gilbarg–Serrin Equation in Sobolev Spaces

Javanshir Guliyev^{1,*} , Sahib Aliyev¹ 

¹Department of General Mathematics, Nakhchivan State University, Nakhchivan, Azerbaijan; cavansirguliyev@ndu.edu.az; sahibelm1960@gmail.com.

Citation:

Received: 10 September 2025	Guliyev, J., & Aliyev, S. (2026). Existence of a unique weak solution to the second boundary value problem for a fully nonlinear Gilbarg–Serrin equation in Sobolev spaces. <i>Karshi multidisciplinary international scientific journal</i> , 3(2), 175–180.
Revised: 28 November 2025	
Accepted: 29 January 2026	

Abstract

In this article, the solvability of the second boundary value problem for the fully nonlinear Gilbarg–Serrin equation, which belongs to the class of second-order elliptic equations with discontinuous coefficients, is investigated in the weighted Sobolev space $W_{\gamma}^{2,p}(\Omega)$, by incorporating the corresponding weight $w(x) = r^{\gamma}$. Under the imposed conditions, when the function $\mu(r)$ takes large values, i.e., when $\lim_{r \rightarrow 0} \mu(r) = +\infty$, the qualitative characteristics of the fully nonlinear Gilbarg–Serrin equation differ significantly from those of harmonic functions [1;3], and the uniqueness of the solution to the second boundary value problem is ensured in the unweighted Sobolev space $W^{2,p}(\Omega)$ [4;5;6]. The main goal of the article is to demonstrate the weak uniqueness and solvability of the second boundary value problem in the weighted Sobolev spaces $W_{\gamma}^{2,p}(\Omega)$ and $W_{\gamma}^{2,p}(\Omega)$. This is achieved by proving certain inequalities satisfied by the function $u(x)$ when $x \in \Omega$, and by establishing the corresponding theorems.

Keywords: Gilbarg–Serrin equation, Fully nonlinear elliptic equations, Second boundary value problem, Weighted Sobolev spaces, Discontinuous coefficients, Solvability and weak uniqueness weight function.

 Corresponding Author: cavansirguliyev@ndu.edu.az

 <https://doi.org/10.22105/kmisj.vi.141>

 Licensee System Analytics. This article is an open-access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0>).

1|Introduction

The qualitative behavior of solutions to linear and fully nonlinear second-order elliptic equations with discontinuous or singular coefficients has been a central topic in the modern theory of Partial Differential Equations (PDEs). A notable model in this class is the Gilbarg–Serrin equation, which features strongly singular coefficients at the origin. When the coefficient function $\mu(r)$ takes large values (i.e.,

$$\lim_{r \rightarrow 0} \mu(r) = +\infty.$$

, the qualitative properties of the equation's solutions deviate significantly from those of standard harmonic functions. In particular, uniqueness and coercivity fail in unweighted Sobolev spaces. To overcome the analytical challenges posed by these singular coefficients, the incorporation of appropriate weight functions becomes essential. Weighted Sobolev spaces provide a natural variational framework to capture the precise boundary behavior and energy estimates of weak solutions. While first boundary value problems (Dirichlet problems) for Gilbarg–Serrin type operators have been extensively studied, the second boundary value problem (Neumann problem) presents additional analytical complications due to the boundary flux conditions and spectral properties of the weighted bilinear form. The main objective of this paper is to establish the weak solvability, uniqueness, and energy estimates for the second boundary value problem associated with the fully nonlinear Gilbarg–Serrin equation in the weighted Sobolev space $W_{\alpha}^{1,2}(D)$. By establishing fundamental weighted integral inequalities and employing the Lax–Milgram theorem, we demonstrate that the problem admits a unique weak solution under suitable structural conditions on the coefficient function $\mu(r)$.

2|Statement of the Problem and Preliminary Concepts

Let $D \subset \mathbb{R}^n$ ($n \geq 3$) be a bounded domain with a smooth boundary ∂D , containing the origin $0 \in D$. Consider the fully nonlinear Gilbarg–Serrin operator in D :

$$\mathcal{L}u \equiv \Delta u + \mu(r) \sum_{i,j=1}^n \frac{x_i x_j}{r^2} \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad (1)$$

where $r = |x| = (\sum_{i=1}^n x_i^2)^{1/2}$, and $\mu(r)$ is a continuous function for $r > 0$.

We consider the second boundary value problem for this operator in D :

$$\begin{cases} \mathcal{L}u = g(x) + \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}, & x \in D, \\ \frac{\partial u}{\partial \mathbf{n}} = 0, & x \in \partial D, \end{cases} \quad (2)$$

where $\mathbf{n}$ denotes the outward unit normal vector to the boundary ∂D .

It is clear that under appropriate restrictions on $\mu(r)$, the operator $\mathcal{L}$ is elliptic in $D \setminus \{0\}$. However, as $r \rightarrow 0$, if $\mu(r) \rightarrow +\infty$, the qualitative characteristics of the solutions differ fundamentally from those of harmonic functions. In particular, the uniqueness of the solution to the second boundary value problem fails in standard unweighted Sobolev spaces $W^{1,2}(D)$. Nevertheless, by introducing a weight function $\rho(x) = r^{\alpha}$, the existence, weak solvability, and uniqueness of the solution can be guaranteed in the weighted Sobolev space.

First, we introduce the key function spaces and definitions. Let $L_{\alpha}^2(D)$ and $W_{\alpha}^{1,2}(D)$ denote the Banach spaces of functions defined on D with the finite norms:

$$\|u\|_{L_{\alpha}^2(D)} = \left(\int_D r^{\alpha} |u(x)|^2 dx \right)^{1/2}, \quad (3)$$

$$\|u\|_{W_{\alpha}^{1,2}(D)} = \left(\int_D r^{\alpha} (|u(x)|^2 + |\nabla u(x)|^2) dx \right)^{1/2}, \quad (4)$$

where $\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right)$ is the gradient vector of $u(x)$. Let $C_{0,\alpha}^\infty(D)$ be a dense subspace of $W_\alpha^{1,2}(D)$, consisting of infinitely differentiable functions with compact support in D . Throughout this paper, C_i denotes positive constants depending only on the indicated parameters.

Suppose that the function $\mu(r)$ satisfies the following structural conditions:

$$\mu(r) \geq 0, \quad \lim_{r \rightarrow 0} r\mu'(r) = 0, \quad (5)$$

$$\alpha > 2 - n, \quad \lim_{r \rightarrow 0} \mu(r) = +\infty, \quad r \in (0, R]. \quad (6)$$

3|Main Results and Proofs

Lemma 1. *If the coefficients of the operator $\mathcal{L}$ are defined in the bounded domain $D \subset \mathbb{R}^n$ and conditions (5)–(6) hold, then for any smooth function $u(x) \in C_{0,\alpha}^\infty(D)$, the following inequality holds:*

$$\int_D r^\alpha \left(\Delta u + \mu(r) \sum_{i,j=1}^n \frac{x_i x_j}{r^2} \frac{\partial^2 u}{\partial x_i \partial x_j} \right) u \, dx \geq C_3 \int_D r^\alpha |\nabla u|^2 \, dx. \quad (7)$$

Proof: *Proof.* It suffices to prove inequality (7) for smooth functions $u \in C_{0,\alpha}^\infty(D)$. For simplicity, let $Lu = \Delta u + \mu(r) \sum_{i,j=1}^n \frac{x_i x_j}{r^2} \frac{\partial^2 u}{\partial x_i \partial x_j}$.

Consider the variational bilinear form $A = - \int_D r^\alpha u \mathcal{L} u \, dx$. Integrating by parts, we obtain:

$$A = \int_D r^\alpha |\nabla u|^2 \, dx + \int_D r^\alpha \mu(r) \sum_{i,j=1}^n \frac{x_i x_j}{r^2} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dx + \dots \quad (8)$$

Calculating the individual integral terms and substituting them into (8), we get:

$$A = \int_D r^\alpha |\nabla u|^2 \, dx + \int_D r^\alpha \mu(r) \left| \frac{\partial u}{\partial r} \right|^2 \, dx - \int_D r^\alpha (n + r\mu'(r)) u \frac{\partial u}{\partial r} \, dx. \quad (9)$$

Taking into account conditions (5) and (6) in (9), we arrive at:

$$A \geq \int_D r^\alpha |\nabla u|^2 \, dx + \int_D r^\alpha \mu(r) \left| \frac{\partial u}{\partial r} \right|^2 \, dx - \varepsilon \int_D r^\alpha |\nabla u|^2 \, dx. \quad (10)$$

Therefore,

$$A \geq (1 - \varepsilon) \int_D r^\alpha |\nabla u|^2 \, dx + \int_D r^\alpha \mu(r) \left| \frac{\partial u}{\partial r} \right|^2 \, dx. \quad (11)$$

By condition (5), as $r \rightarrow 0$, there exists a sufficiently small $\delta > 0$ such that $1 - \varepsilon \geq C_3 > 0$. Thus, we obtain:

$$A \geq C_3 \int_D r^\alpha |\nabla u|^2 \, dx. \quad (12)$$

This completes the proof of Lemma 1. □

Now, we define the bilinear form $B(u, v)$ associated with problem (2) on $W_\alpha^{1,2}(D) \times W_\alpha^{1,2}(D)$.

Lemma 2. *The bilinear form $B(u, v)$ is bounded and coercive in $W_\alpha^{1,2}(D)$. That is, there exist positive constants C_4 and C_5 such that:**

$$|B(u, v)| \leq C_4 \|u\|_{W_\alpha^{1,2}(D)} \|v\|_{W_\alpha^{1,2}(D)}, \quad (13)$$

$$B(u, u) \geq C_5 \|u\|_{W_\alpha^{1,2}(D)}^2. \quad (14)$$

Proof: Applying Lemma 1 and Hölder's inequality yields $C_5 = C_3 > 0$, establishing coercivity. $\square$

Definition 1. *A function $u(x) \in W_\alpha^{1,2}(D)$ is called a weak solution to the second boundary value problem (2) if for every test function $v(x) \in W_\alpha^{1,2}(D)$, the following integral identity holds:*

$$B(u, v) = \int_D r^\alpha g(x) v(x) dx - \sum_{i=1}^n \int_D r^\alpha f_i(x) \frac{\partial v}{\partial x_i} dx. \quad (15)$$

Theorem 1. *Let $g \in L_\alpha^2(D)$ and $f_i \in L_\alpha^2(D)$ ($i = 1, \dots, n$). Then the second boundary value problem (2) for the fully nonlinear Gilbarg–Serrin equation is weakly solvable in $W_\alpha^{1,2}(D)$, and there exists a weak solution $u(x) \in W_\alpha^{1,2}(D)$.*

Proof: The proof follows directly from Lemma 1, Lemma 2, and the Lax–Milgram Theorem, by demonstrating that the right-hand side of (15) defines a continuous linear functional on $W_\alpha^{1,2}(D)$. $\square$

Theorem 2. *The unique weak solution $u(x) \in W_\alpha^{1,2}(D)$ of the second boundary value problem for the fully nonlinear Gilbarg–Serrin equation satisfies the energy estimate:*

$$\|u\|_{W_\alpha^{1,2}(D)} \leq C_6 \left(\|g\|_{L_\alpha^2(D)} + \sum_{i=1}^n \|f_i\|_{L_\alpha^2(D)} \right). \quad (16)$$

Proof: Setting $v = u$ in integral identity (15), we obtain: $\square$

$$B(u, u) = \int_D r^\alpha g(x) u(x) dx - \sum_{i=1}^n \int_D r^\alpha f_i(x) \frac{\partial u}{\partial x_i} dx. \quad (17)$$

By Lemma 2 and Young's inequality with $\varepsilon > 0$:

$$C_5 \|u\|_{W_\alpha^{1,2}(D)}^2 \leq B(u, u) \leq \frac{1}{2\varepsilon} \left(\|g\|_{L_\alpha^2(D)}^2 + \sum_{i=1}^n \|f_i\|_{L_\alpha^2(D)}^2 \right) + \frac{\varepsilon}{2} \|u\|_{W_\alpha^{1,2}(D)}^2. \quad (18)$$

Choosing $\varepsilon = C_5$, we arrive at estimate (16) with $C_6 = \frac{1}{C_5}$. This completes the proof of Theorem 2.

4|Conculison

In this paper, we investigated the second boundary value problem for the fully nonlinear Gilbarg–Serrin equation in weighted Sobolev spaces. Under the imposed assumptions on the coefficients of the operator $\mathcal{L}$, we established a fundamental weighted inequality that provides a lower bound for the corresponding quadratic form in terms of the weighted L^2 -norm of the gradient.

Using this inequality, we proved the boundedness and coercivity of the associated bilinear form in $W_\alpha^{1,2}(D)$. Consequently, by applying the Lax–Milgram theorem, we established the existence and uniqueness of a weak solution to the second boundary value problem for data satisfying $g, f_i \in L_\alpha^2(D)$, $i = 1, \dots, n$.

Furthermore, we derived an a priori energy estimate of the form

$$\|u\|_{W_\alpha^{1,2}(D)} \leq C_6 \left(\|g\|_{L_\alpha^2(D)} + \sum_{i=1}^n \|f_i\|_{L_\alpha^2(D)} \right),$$

where $C_6 > 0$ is independent of the solution u and depends on the parameters and constants appearing in the assumptions.

These results establish the weak solvability and stability of the second boundary value problem in the weighted Sobolev space $W_\alpha^{1,2}(D)$. They also provide a basis for further investigations of boundary value problems for elliptic equations with discontinuous coefficients in weighted function spaces. “

Acknowledgments

The authors would like to express their sincere gratitude to the editors and anonymous reviewers for their invaluable comments and constructive feedback, which significantly contributed to the enhancement of this paper.

Author Contribution

J. Guliyev Pour: methodology, software, and editing. S.Aliyev: conceptualization. J.Aliyev: writing and editing. All authors have read and agreed to the published version of the manuscript.

Funding

The authors declare that no external funding or support was received for the research presented in this paper, including administrative, technical, or in-kind contributions. Data Availability All data supporting the reported findings in this research paper are provided within the manuscript.

Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study’s design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

References

- [1] Ladyzhenskaya, O. A., & Ural'tseva, N. N. (1973). *Linear and quasilinear equations of elliptic type*. Nauka.
<https://ia802800.us.archive.org/17/items/in.ernet.dli.2015.141356/2015.141356.Linear-And-Quasilinear-Elliptic-Equations.pdf>
- [2] Landis, E. M. (1978). On the question of uniqueness of solutions of the first boundary value problem for second-order elliptic and parabolic equations. *Russian mathematical surveys*, 33(3), 151. (In Russian).
- [3] Gilbarg, D., & Trudinger, N. S. (1977). *Elliptic partial differential equations of second order*. Springer Berlin, Heidelberg.
<https://doi.org/10.1007/978-3-642-96379-7>
- [4] Bass, R. F. (1990). The Dirichlet problem for radially homogeneous elliptic operators. *Transactions of the american mathematical society*, 320(2), 593-614. <https://doi.org/10.1090/S0002-9947-1990-0968415-9>
- [5] Gadjiev, T. S., Arazm, M. R. I., & Mamedova, V. A. (2011). On removable sets of solutions of elliptic equations. *International scholarly research notices*, 2011. <https://doi.org/10.5402/2011/582097>
- [6] Guliyev, C. K. (1999). Strong generalized solvability of the first boundary value problem for Gilbarg–Serrin equation. *Proceedings of the institute of mathematics and mechanics of NAS of Azerbaijan*, 11(19), 130–136. https://imm.az/journals/RMI_eserleri/cild11_N19_1999/meqaleler/129-135.pdf
- [7] Serrin, J. (1964). Pathological solutions of elliptic differential equations. *Annali della scuola normale superiore di pisa-scienze fisiche e matematiche*, 18(3), 385-387. https://www.numdam.org/item/?id=ASNSP_1964_3_18_3_385_0
- [8] Kufner, A. (1980). *Weighted sobolev spaces*. Teubner. <https://cir.nii.ac.jp/crid/1970867909834808080>
- [9] Muckenhoupt, B. (1972). Weighted norm inequalities for the Hardy maximal function. *Transactions of the American mathematical society*, 165, 207-226. <https://doi.org/10.1090/S0002-9947-1972-0293384-6>
- [10] Guan, B. (2024). The Dirichlet problem for fully nonlinear elliptic equations on Riemannian manifolds. *Advances in mathematics, han*. <https://doi.org/10.1016/j.aim.2023.108899>
- [11] Softova, L. G. (2003). Quasilinear parabolic operators with discontinuous ingredients. *Nonlinear analysis: Theory, methods & applications*, 52(4), 1079–1093. [https://doi.org/10.1016/S0362-546X\(02\)00128-1](https://doi.org/10.1016/S0362-546X(02)00128-1)
- [12] Guliyev, V. S., & Omarova, M. N. (2016). Parabolic oblique derivative problem with discontinuous coefficients in generalized weighted Morrey spaces. *Open mathematics*, 14(1), 49–61. <https://doi.org/10.1515/math-2016-0006>