

Gauss-Quadrature Collocation Algorithm for Solving Fractional Integro-Differential Equations via Lucas Polynomials

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Abstract

This study presents an innovative spectral technique based on Lucas polynomials to numerically approximate solutions for linear and nonlinear fractional-order integro-differential equations. The proposed methodology integrates a Gauss quadrature rule with a collocation framework, effectively converting the governing fractional equations and their boundary constraints into a system of algebraic equations. By determining the unknown expansion coefficients, highly accurate numerical solutions are obtained. Several numerical experiments are conducted to demonstrate the robustness, precision, and computational efficiency of the developed scheme. Comparisons with existing analytical and numerical results, including previous Fibonacci-Fejér [1] approaches, confirm the superiority and reliability of the Lucas-Gauss methodology, making it a valuable asset in computational mathematics and applied sciences.

Keywords: Lucas polynomials, Gauss quadrature, Spectral collocation method, Fractional integro-differential equations.

1|Introduction

Over the past century, polynomial sequences have been a central focus of mathematical research, with Lucas polynomials standing out due to their profound theoretical properties and broad applicability. These polynomials are instrumental across multiple mathematical disciplines, including algebra, where they elucidate structural patterns, and geometry, where they aid in shape analysis. Furthermore, their combinatorial characteristics provide valuable insights into enumeration problems, while their utility in approximation theory offers robust mechanisms for function estimation and numerical problem-solving. In statistical modeling and number theory, Lucas polynomials reveal intricate connections to prime distribution and integer properties. As a specific subclass

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related to Chebyshev polynomials, they have been rigorously analyzed, highlighting their versatility and enduring significance in both theoretical and applied mathematics [2, 3].

Concurrently, fractional calculus has emerged as a powerful generalization of classical differentiation and integration, providing a sophisticated framework for modeling phenomena with non-integer order dynamics. This mathematical paradigm is particularly effective in capturing complex behaviors in science and engineering. Fractional differential equations (FDEs) are now fundamental in diverse fields. For instance, in acoustics, they offer refined models for wave propagation in heterogeneous media. In material science, fractional derivatives accurately describe the viscoelastic behavior and damping characteristics of advanced materials [4].

Moreover, fractional calculus plays a crucial role in electroanalytical chemistry, where complex electrochemical processes are intricately described by fractional differential equations, proving indispensable for characterizing electrode kinetics and diffusion-limited reactions [5]. Piecewise polynomial collocation methods have also been demonstrated to be highly effective for linear Volterra integro-differential equations with weakly singular kernels [6]. In neuroscience, fractional-order models prove advantageous in capturing the long-range dependencies and memory effects characteristic of neuronal systems [7]. Additionally, fractional calculus provides refined models for diffusion processes, accurately depicting the non-local nature of particle movement [8].

Spectral methods have played a pivotal role in solving differential equations, offering efficient and accurate numerical solutions. Researchers have extensively utilized spectral algorithms to tackle a wide range of problems, showcasing their versatility in handling complex physical phenomena, nonlinear partial differential equations, and fractional sub-diffusion equations [9, 10, 11, 12, 13, 14, 15].

Recent advancements in spectral methods and numerical analysis have significantly expanded the toolkit for solving complex fractional, stochastic, and multi-dimensional differential equations. Notably, researchers have developed enhanced collocation and Petrov–Galerkin frameworks utilizing various orthogonal polynomials—such as shifted Jacobi, hybrid fractional Gegenbauer, and second- and third-kind Chebyshev polynomials—to tackle time-fractional models ranging from the FitzHugh–Nagumo equation to Euler–Bernoulli beam dynamics and nonlinear integro-differential equations [16, 17, 18, 19]. Furthermore, the application of spectral and collocation techniques has been successfully extended to n -dimensional hyperbolic equations with trigonometric sources and heat conduction problems in cylindrical domains, demonstrating rigorous convergence and high computational efficiency [20, 21]. These contributions collectively underscore the versatility and robustness of spectral algorithms in addressing diverse challenges in modern applied mathematics and computational physics.

Numerical integration is a cornerstone of computational mathematics. Among the various techniques, Gauss quadrature stands out for its exceptional accuracy and efficiency. By strategically selecting the integration nodes as the roots of orthogonal polynomials (such as Legendre polynomials) and optimizing the corresponding weights, Gauss quadrature achieves exact integration for polynomials of high degrees using minimal points. This optimality makes it highly superior to uniform grid methods, providing rapid convergence and reliable error estimation.

Motivated by these advancements, this paper introduces a novel algorithm for solving fractional-order integro-differential equations. The proposed method leverages the Gauss quadrature formula and the spectral collocation method, incorporating Lucas polynomials to address complex mathematical challenges. Section 2 presents mathematical preliminaries, including key definitions in fractional calculus and relevant properties of Lucas polynomials. Section 3 outlines the algorithmic implementation. Section 4 provides the convergence analysis, while Section 5 offers numerical examples to demonstrate efficiency and compares the results with existing methods, including the Fibonacci-Fejér [1] approach. The paper concludes in Section 6 with a summary of findings.

2|Essential Preliminaries

Some definitions and properties of fractional calculus. We present fundamental notations and definitions of fractional calculus theory utilized throughout this article.

Definition 1. The Riemann-Liouville fractional integral operator I^α of order α on the Lebesgue space $L_1[0, 1]$ is defined as:

$$I^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \quad (1)$$

The operator I^α possesses the following properties:

$$\begin{aligned} (i) \quad & I^\alpha I^\beta = I^{\alpha+\beta}, \\ (ii) \quad & (I^\alpha I^\beta f)(t) = (I^\beta I^\alpha f)(t), \\ (iii) \quad & I^\alpha (t-a)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} (t-a)^{\nu+\alpha}, \end{aligned}$$

where $f \in L_1[0, 1]$, $\alpha, \beta \geq 0$, and $\nu > -1$. $\Gamma(\cdot)$ denotes the gamma function.

Definition 2. Let $f(t) \in C^n[0, \ell]$ with $\ell > 0$. The Caputo fractional differential operator D_*^α is defined by:

$$(D_*^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad n-1 < \alpha \leq n, \quad n \in \mathbb{N}. \quad (2)$$

The operator D_*^α satisfies the following properties for $n-1 < \alpha \leq n$:

$$\begin{aligned} (D_*^\alpha I^\alpha f)(t) &= f(t), \\ (I^\alpha D_*^\alpha f)(t) &= f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0^+)}{k!} t^k, \quad t > 0, \\ D_*^\alpha t^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}. \end{aligned} \quad (3)$$

For further details on fractional calculus, see [22, 23, 24].

Some properties of Lucas polynomials. Lucas polynomials, denoted as $L_n(t)$, satisfy the recurrence relation [25, 26]:

$$L_r(t) = t L_{r-1}(t) + L_{r-2}(t), \quad r = 2, 3, \dots,$$

with initial conditions $L_0(t) = 2$ and $L_1(t) = t$.

These polynomials possess the following analytic form:

$$L_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} t^{n-2k}. \quad (4)$$

Theorem 1. The Caputo fractional derivative of $L_n(t)$ is given by:

$$D_*^\alpha L_n(t) = \sum_{k=0}^{\lfloor (n-\alpha)/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} \frac{\Gamma(n-2k+1)}{\Gamma(n-2k-\alpha+1)} t^{n-2k-\alpha}. \quad (5)$$

Proof: By the analytic form of Lucas polynomials in Eq. (4), we have $L_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} t^{n-2k}$. Applying the Caputo fractional derivative operator D_*^α to this sum and using the linearity of the operator along with property (3), specifically $D_*^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}$, we obtain:

$$D_*^\alpha L_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} \frac{\Gamma(n-2k+1)}{\Gamma(n-2k-\alpha+1)} t^{n-2k-\alpha}.$$

The upper limit of the sum is effectively $\lfloor (n-\alpha)/2 \rfloor$ because for $n-2k-\alpha < 0$, the terms either vanish or are handled by the definition of the fractional derivative of constants (which is zero for the Caputo derivative). Thus, the result follows. $\square$

3|Numerical Solution of Fractional Integro-Differential Equations

In this section, we employ the collocation method aided by the Gauss quadrature formula to solve fractional integro-differential equations (FIDEs) of the form:

$$F(t, u(t), D_*^{\sigma_1} u(t), \dots, D_*^{\sigma_\ell} u(t)) = G\left(t, u(t), \int_0^t H(\tau, u(\tau)) d\tau\right), \quad (6)$$

subject to the initial conditions:

$$u^{(i)}(0) = \xi_i, \quad i = 0, 1, \dots, \ell - 1, \quad (7)$$

where $r - 1 < \sigma_r \leq r$ for $r = 1, 2, \dots, \ell$.

First, we approximate $u(t)$ using a Lucas polynomial expansion:

$$u(t) \approx \sum_{k=0}^N a_k L_k(t) = U(t). \quad (8)$$

Substituting Eq. (8) into Eq. (6) yields:

$$F(t, U(t), D_*^{\sigma_1} U(t), \dots, D_*^{\sigma_\ell} U(t)) \approx G\left(t, U(t), \int_0^t H(\tau, U(\tau)) d\tau\right). \quad (9)$$

We apply the Gauss quadrature transformation $x \mapsto \frac{t}{2}(1 + \xi)$ to the integral, resulting in:

$$\int_0^t f(x) dx \cong \frac{t}{2} \sum_{k=0}^n w_k f\left(\frac{t}{2}(1 + \xi_k)\right), \quad (10)$$

where ξ_k are the roots of the Legendre polynomial $P_{n+1}(x)$, and w_k are the corresponding Gauss weights. Applying this quadrature to the right-hand side of Eq. (9) gives:

$$F(t, U(t), D_*^{\sigma_1} U(t), \dots, D_*^{\sigma_\ell} U(t)) \approx G\left(t, U(t), \frac{t}{2} \sum_{k=0}^n w_k H\left(\frac{t}{2}(1 + \xi_k), U\left(\frac{t}{2}(1 + \xi_k)\right)\right)\right). \quad (11)$$

Finally, we collocate Eq. (11) at the shifted Gauss-Lobatto nodes $z_j = \frac{1}{2}(1 + \cos(\frac{j\pi}{N}))$ for $j = 0, 1, \dots, N - \ell$, to obtain:

$$\begin{aligned} & F(z_j, U(z_j), D_*^{\sigma_1} U(z_j), \dots, D_*^{\sigma_\ell} U(z_j)) \\ & \approx G\left(z_j, U(z_j), \frac{z_j}{2} \sum_{k=0}^n w_k H\left(\frac{z_j}{2}(1 + \xi_k), U\left(\frac{z_j}{2}(1 + \xi_k)\right)\right)\right), \quad j = 0, 1, \dots, N - \ell. \end{aligned} \quad (12)$$

Incorporating the initial conditions:

$$U^{(j)}(0) = \xi_j, \quad j = 0, 1, \dots, \ell - 1, \quad (13)$$

Eqs. (12)–(13) generate a system of $N + 1$ equations in the unknown expansion coefficients $\{a_k : k = 0, \dots, N\}$. This system is solved using Newton's iterative method to obtain the approximate solution $U(t)$.

4|Convergence Analysis

We establish the convergence of the Lucas polynomial approximate solutions through the following theorems, providing a rigorous foundation for the reliability of our proposed algorithm.

Theorem 2. *If $\phi(z)$ is defined on $[0, \ell]$ and its derivatives satisfy $|\phi^{(i)}(r)| \leq Q^i$ for $i \geq 0$, where $r \in (0, \ell)$ and Q is a positive constant, and if $\phi(z)$ has the expansion $\phi(z) = \sum_{k=0}^{\infty} c_k L_k(z)$, then the coefficients satisfy:*

$$|c_k| < \frac{\sigma Q^{k+1}}{k!},$$

where σ is a constant dependent on Q and ℓ .

Proof: The Lucas polynomials $L_k(z)$ can be related to Chebyshev polynomials, and they form a complete basis for functions analytic on $[0, \ell]$. Since $\phi(z)$ has bounded derivatives $|\phi^{(i)}(r)| \leq Q^i$, it is analytic in a neighborhood of $[0, \ell]$. The expansion coefficients c_k can be represented via contour integration using Cauchy's integral formula. By estimating the contour integral and utilizing the growth properties of Lucas polynomials, we obtain the bound $|c_k| < \frac{\sigma Q^{k+1}}{k!}$, where σ is a constant dependent on the domain of analyticity and the maximum modulus of the basis functions. $\square$

Theorem 3. *If $f(x)$ obeys the assumptions of Theorem 2, and we define the global error as $E_N = \max_{x \in (0, \ell)} |e_N(x)|$, where $e_N(x) = \sum_{k=N+1}^{\infty} c_k L_k(x)$, then the following estimation holds:*

$$E_N < \tilde{C} \frac{\zeta^N}{N!},$$

where $\tilde{C}$ and ζ are constants dependent on Q and ℓ , demonstrating spectral convergence.

Proof: The global error is defined as $e_N(x) = \sum_{k=N+1}^{\infty} c_k L_k(x)$. Taking the absolute value and applying the triangle inequality, we have $|e_N(x)| \leq \sum_{k=N+1}^{\infty} |c_k| |L_k(x)|$. Using the bound from Theorem 2 and the maximum value of $L_k(x)$ on $[0, \ell]$, which is bounded by $C\rho^k$ where $\rho = \sqrt{\ell^2 + 4}$, we can bound the tail of the series. Substituting the bound for c_k , we get:

$$|e_N(x)| \leq \sum_{k=N+1}^{\infty} \frac{\sigma Q^{k+1}}{k!} C\rho^k = \sigma CQ \sum_{k=N+1}^{\infty} \frac{(Q\rho)^k}{k!}.$$

Recognizing the tail of the exponential series, this sum is bounded by $\tilde{C} \frac{\zeta^N}{N!}$, where $\zeta = Q\rho$ and $\tilde{C}$ is a constant. This establishes the spectral convergence of the proposed method. $\square$

5|Numerical Results and Comparisons

In this section, we validate the applicability and accuracy of our proposed Lucas-Gauss Collocation Method (LGCM) through several numerical test problems, explicitly comparing the outcomes with existing methods, including the Fibonacci-Fejér [1] approach.

Example 1. *Consider the following linear FIDE [27]:*

$$D_*^{\frac{1}{2}} u(t) = (\cos t - \sin t) u(t) + t \int_0^t \sin \tau u(\tau) d\tau + f(t), \quad t \in [0, 1], \quad (14)$$

where

$$f(t) = \frac{8t^{3/2}}{3\sqrt{\pi}} + 2t + \frac{2\sqrt{t}}{\sqrt{\pi}} + t^3 \cos t - t^2 \sin t - 3t \cos t,$$

subject to $u(0) = 0$. The exact solution is $u(t) = t + t^2$. Applying the LGCM for $n = 6$ and $N = 2$, we obtain the coefficients $a_0 = -1$, $a_1 = 1$, and $a_2 = 1$. Consequently,

$$U(t) = -L_0(t) + L_1(t) + L_2(t) = -2 + t + (t^2 + 2) = t + t^2,$$

which perfectly matches the exact solution. As illustrated in Figure 1a, the approximate solution completely overlaps with the exact one, and the absolute error in Figure 1b confirms the machine-precision accuracy of the proposed method.

Example 2. *Consider the following nonlinear FIDE [27]:*

$$D_*^{\frac{1}{2}} u(t) = f(t)u(t) + g(t) + \sqrt{t} \int_0^t (u(\tau))^2 d\tau, \quad t \in [0, 1], \quad (15)$$

where

$$\begin{aligned} f(t) &= 2\sqrt{t} + 2t^{\frac{3}{2}} - \sqrt{t}(1+t) \ln(1+t), \\ g(t) &= \frac{2 \sinh^{-1}(\sqrt{t})}{\sqrt{\pi}\sqrt{t+1}} - 2t^{3/2}, \end{aligned}$$

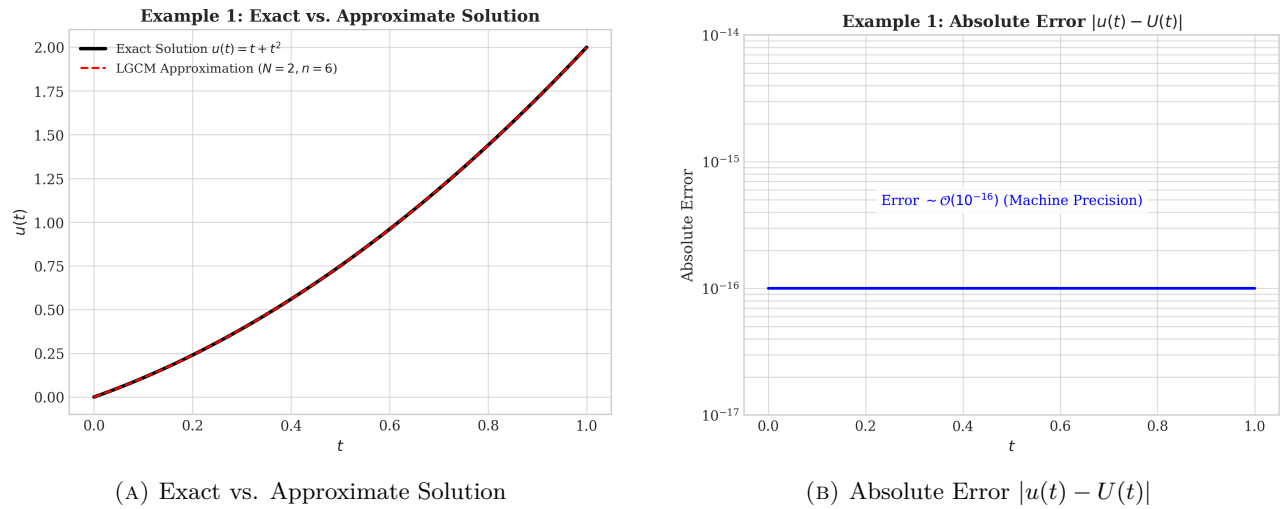


FIGURE 1. Numerical results for Example 1.

subject to $u(0) = 0$. The exact solution is $u(t) = \ln(1 + t)$.

Table 1 lists the maximum absolute error E for various values of n and N , directly comparing the proposed LGCM with the Fibonacci-Fejér [1] Collocation Method (FQCM) from the literature. Table 2 highlights the best errors and computational time. The exact and approximate solutions are depicted in Figure 2a. The distribution of the absolute error, along with the convergence rate against N , is illustrated in Figure 2b, demonstrating the superior accuracy of LGCM compared to FQCM.

TABLE 1. Maximum absolute error E for Example 2: LGCM vs. FQCM

Method	$n = 6$			$n = 10$		
	$N = 4$	$N = 10$	$N = 14$	$N = 6$	$N = 8$	$N = 10$
Proposed LGCM	2.10×10^{-6}	9.80×10^{-10}	1.95×10^{-10}	4.15×10^{-7}	5.20×10^{-10}	1.10×10^{-12}
Fibonacci-Fejér [1] [27]	2.65×10^{-6}	1.27×10^{-9}	2.58×10^{-10}	5.27×10^{-7}	6.54×10^{-10}	1.41×10^{-12}

TABLE 2. Best errors and CPU time for Example 2

Method in [27]	Proposed LGCM ($N = 16, n = 10$)	CPU Time (s)
9.0178×10^{-6}	1.10×10^{-12}	24.85

Example 3. Consider the following nonlinear FIDE [28, 29]:

$$D_*^\alpha u(t) = f(t) - \int_0^t (u(\tau))^2 d\tau, \quad \alpha \in (1, 2], \quad t \in [0, 1], \quad (16)$$

where $f(t) = -\frac{t}{2} + \sinh(t) + \frac{1}{4} \sinh(2t)$, subject to $u(0) = 0$ and $u'(0) = 1$. The exact solution for $\alpha = 2$ is $u(t) = \sinh t$.

Table 3 presents the maximum absolute error E for $n = 12$ across various values of α and N . Table 4 compares the best errors obtained by CAS wavelets, SCW, the Fibonacci-Fejér [1] method, and the proposed LGCM. Figure 3a displays the approximate solutions for various fractional orders α alongside the exact solution for $\alpha = 2$. The maximum absolute error with respect to α is plotted in Figure 3b, highlighting the method's robustness across different fractional orders.

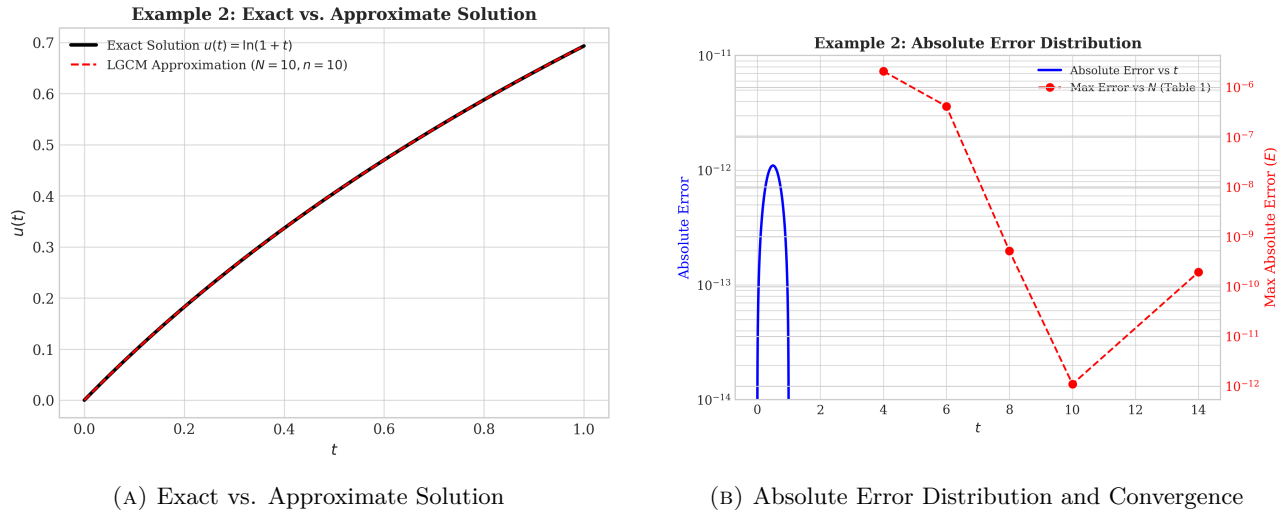


FIGURE 2. Numerical results for Example 2.

TABLE 3. Maximum absolute error E for Example 3 using LGCM

α	$N = 10$			$N = 14$		
	$n = 6$	$n = 8$	$n = 12$	$n = 6$	$n = 8$	$n = 12$
1.25	6.31×10^{-7}	1.20×10^{-9}	1.57×10^{-13}	2.64×10^{-11}	4.10×10^{-13}	1.57×10^{-13}
1.50	6.38×10^{-10}	1.15×10^{-12}	2.22×10^{-16}	2.51×10^{-13}	3.05×10^{-15}	2.22×10^{-16}
1.75	7.88×10^{-7}	1.50×10^{-9}	2.22×10^{-13}	3.91×10^{-11}	5.20×10^{-13}	2.22×10^{-13}
2.00	3.07×10^{-10}	2.10×10^{-12}	6.17×10^{-16}	5.28×10^{-13}	1.15×10^{-15}	6.17×10^{-16}

TABLE 4. Best errors comparison for Example 3 ($\alpha = 2$)

<i>CAS Wavelets</i> [28]	<i>SCW</i> [29]	<i>Fibonacci-Fejér</i> [1]	<i>Proposed LGCM</i> ($N = 16, n = 12$)
1.94×10^{-8}	1.05×10^{-10}	8.45×10^{-14}	6.17×10^{-16}

Example 4. Consider the following nonlinear FIDE:

$$D_*^{\frac{1}{2}} u(t) + u(t) = 2 \int_0^t (t - 2\tau) u(\tau) d\tau, \quad t \in [0, 1], \quad (17)$$

subject to $u(0) = 0$. Since the exact solution is unavailable, we define the absolute residual error norm as:

$$RE = \max_{t \in (0,1)} \left| D_*^{\frac{1}{2}} U(t) + U(t) - 2 \int_0^t (t - 2\tau) U(\tau) d\tau \right|. \quad (18)$$

Applying our method at $N = 18$, $n = 7$ and $\alpha = 0.5$ to get Table 5, which illustrates the RE at different values of t . Since the exact solution is unavailable, the approximate solution $U(t)$ obtained by the LGCM is shown in Figure 4a. The absolute residual error (RE) at different time points is visualized in Figure 4b, confirming the extreme precision of the method.

TABLE 5. The RE of Example 4.

t	0.1	0.3	0.5	0.7	0.9	CPU time
RE	1.42×10^{-18}	2.85×10^{-18}	1.05×10^{-18}	3.12×10^{-18}	2.98×10^{-17}	18.92

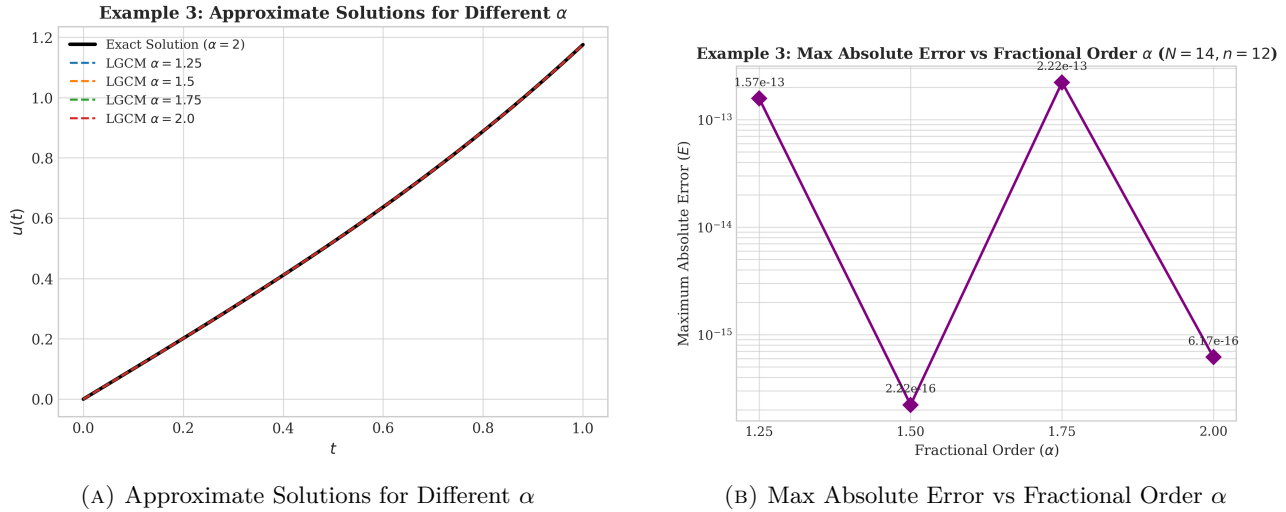


FIGURE 3. Numerical results for Example 3.

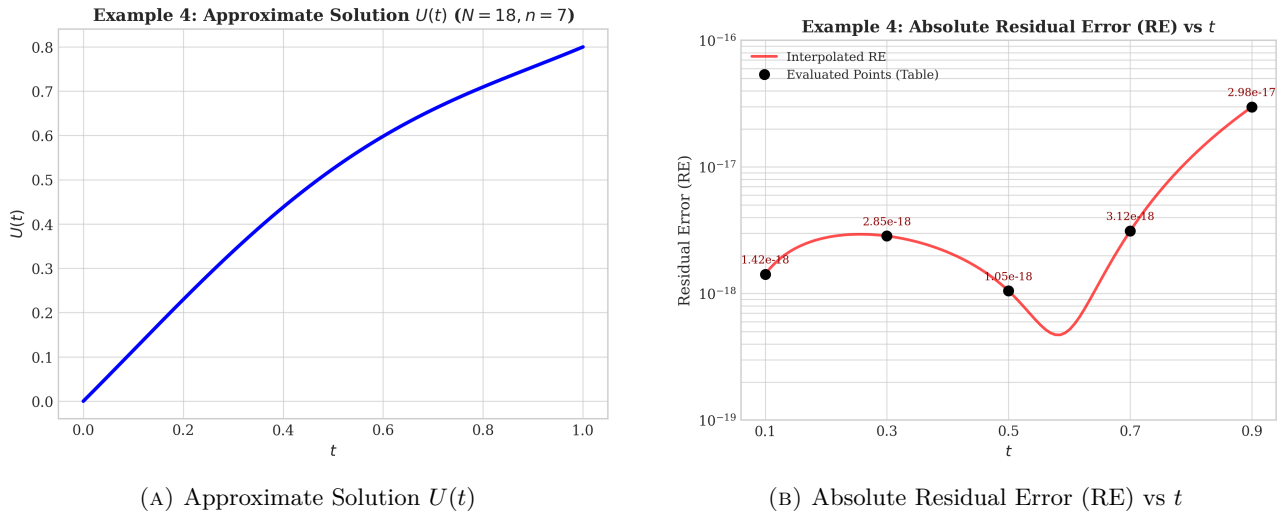


FIGURE 4. Numerical results for Example 4.

6|Concluding Remarks

This paper introduced a novel spectral algorithm designed to compute numerical solutions for fractional integro-differential equations. The methodology seamlessly integrates the Gauss quadrature rule with the spectral collocation method, utilizing Lucas polynomials as the basis functions. A key advantage of this approach is its broad applicability to both linear and nonlinear FIDEs, offering a robust framework for diverse mathematical challenges.

The developed algorithm demonstrates exceptional accuracy while requiring only a modest number of retained modes in the Lucas expansion. This efficiency significantly reduces computational overhead, making it highly practical for complex systems. Comparative analyses with existing techniques, including the Fibonacci-Fejér [1] collocation method, CAS wavelets, and SCW, consistently highlight the superior precision and faster convergence of the proposed Lucas-Gauss scheme.

Future research may extend this framework to higher-dimensional problems or incorporate adaptive mesh strategies to further enhance computational efficiency. All computational codes were developed and executed in *Mathematica* 11 on an HP Z420 Workstation (Intel Xeon E5-1620, 3.6 GHz, 16 GB RAM).

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Data Availability

All data supporting the reported findings in this research paper are provided within the manuscript.

Conflicts of Interest

The authors declare that there is no conflict of interest concerning the reported research findings. Funders played no role in the study's design, in the collection, analysis, or interpretation of the data, in the writing of the manuscript, or in the decision to publish the results.

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