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Exploring the Mathematical Framework of the Gamma Function

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Abstract

The gamma function extends the factorial to a continuous and complex-variable setting and provides a foundation for numerous special functions. This paper examines its basic properties, related functions, and incomplete forms before investigating selected applications to definite integration, statistical mechanics, probability theory, and fractional calculus. Particular attention is given to the conditions under which integral representations and identities are valid. Classical results are connected with recent work on series representations and Stirling-type approximations. The study emphasizes the gamma function both as an analytical object and as a unifying tool in mathematical physics.

Keywords: Special functions, Gamma function, Factorial generalization, Beta functions.

1|Introduction and Preliminaries

The gamma function provides a fundamental connection between discrete mathematics, continuous analysis, and the mathematical formulation of physical phenomena (see, e.g., [1, 2, 3, 4, 5]). Its familiar interpretation as an extension of the factorial is only the starting point of its significance: the function also supplies integral representations, normalization factors, and series coefficients that recur throughout the theory of special functions. Consequently, understanding the gamma function is valuable not only for evaluating a particular class of integrals but also for explaining the mathematical structure of functions used in physical applications. Recent investigations

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into its series representations and numerical approximations demonstrate that this classical function remains an active subject of mathematical research (see, e. g. [6, 7, 8, 9]).

Further, the gamma function is a central special function linking factorials, integral transforms, probability distributions, and mathematical models of physical phenomena. Its importance extends beyond its interpretation as a generalization of the factorial: it also supplies coefficients, normalization factors, and kernels for related functions and operators [3, 4, 9].

The following three forms are equivalence of the gamma function (see the proof [1, 3])

(1): Infinite Limit (Euler)

$$\Gamma(z) \equiv \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{z(1+z)(2+z)\cdots(n+z)}\, n^z, \quad z \neq 0, -1, -2, -3, ... \tag{1}$$

(2): Definite Integral

The notation $\Gamma(z)$ is due to Legendre [1]. If the real part of the complex number z is strictly positive $\mathrm{Re}(z) > 0$, then the integral

$$\Gamma(z) \equiv \int_0^\infty t^{z-1}e^{-t}\, dt, \quad \mathrm{Re}(z) > 0, \tag{2}$$

converges absolutely, and is known as the *Euler integral of the second kind.*

(3): Weierstrass's definition (Infinite-Product Form)

$$\Gamma(z) \equiv \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n}, \quad z \in \mathbb{C} \setminus \mathbb{Z}_0^-, \tag{3}$$

where $\gamma \approx 0.577216$ is the Euler–Mascheroni constant. Or

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} e^{-\frac{z}{n}} \left(1+\frac{z}{n}\right), \quad z \in \mathbb{C} \setminus \mathbb{Z}_0^-. \tag{4}$$

This paper is an analytical review rather than a claim of new theorems. Its objectives are:

(1) to establish the fundamental identities of the gamma function;

(2) to explain its relationships with other special functions;

(3) to study the incomplete gamma functions and their normalization;

(4) to derive selected integral and physical applications;

(5) to explain its structural role in fractional calculus.

Basic properties

- $\Gamma(1) = 1$,
- $\Gamma(z+1) = z\Gamma(z)$,
- $\Gamma(n) = (n-1)!$ for any positive integer n,
- The Euler's reflection formula is $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}, \qquad z \notin \mathbb{Z}$,
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$,
- $\Gamma(z) = \int_0^1 \left(\log \frac{1}{t}\right)^{z-1} dt$,
- $\Gamma(z) = 2\int_0^\infty t^{2z-1}e^{-t^2}\, dt$,

- $\int_0^\infty t^{\alpha-1}e^{-\lambda t}\,dt = \dfrac{\Gamma(\alpha)}{\lambda^\alpha}$, for $\lambda > 0$,
- Legendre's duplication formula is

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z). \tag{5}$$

- The Stirling's approximation $\Gamma(z) \sim \sqrt{2\pi}z^{z+1/2}e^{-z}$ as $z \to \infty$ in $|\arg(z)| < \pi$. This is precise in the sense that the ratio of the approximation to the true value approaches 1 in the limit as $|z|$ goes to infinity.

2|Functions Related to the Gamma Function

This section explains how the gamma function generates or connects other special functions.

The Beta Function

For $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$,

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt. \tag{6}$$

Its relationship with the gamma function is

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{7}$$

Digamma and Polygamma Functions

Define

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \qquad \psi^{(m)}(z) = \frac{d^m}{dz^m}\psi(z). \tag{8}$$

Differentiating the logarithm of the recurrence relation gives

$$\psi(z+1) = \psi(z) + \frac{1}{z}. \tag{9}$$

For positive integers,

$$\psi(n) = H_{n-1} - \gamma_{\mathrm{E}}, \tag{10}$$

where $H_0 = 0$, $H_n = \sum_{k=1}^{n} k^{-1}$, and γ_{E} is the Euler–Mascheroni constant.

Pochhammer Symbols and Hypergeometric Functions

The rising factorial is

$$(a)_n = a(a+1)\cdots(a+n-1), \qquad (a)_0 = 1. \tag{11}$$

Where the quotient is defined,

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{12}$$

The confluent hypergeometric series is

$${}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!}, \tag{13}$$

with $b \notin \{0,-1,-2,\ldots\}$.

This connection prepares the discussion of Coulomb functions [13, 14, 15].

The Incomplete Gamma Functions

For $a > 0$ and $x \geq 0$, the lower incomplete gamma function is

$$\gamma(a, x) = \int_0^x t^{a-1} e^{-t}\, dt. \tag{14}$$

For $a > 0$ and $x \geq 0$, the upper incomplete gamma function is

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t}\, dt. \tag{15}$$

These restrictions provide a simple real-variable starting point. Extensions to complex arguments require branch conventions and analytic continuation [3, Sec. 8.2] and [10].

Now, we have

For $x > 0$,

$$\frac{\partial \gamma(a, x)}{\partial x} = x^{a-1}\, e^{-x}, \qquad \frac{\partial \Gamma(a, x)}{\partial x} = -x^{a-1}\, e^{-x}. \tag{16}$$

Integration by parts gives

$$\gamma(a + 1, x) = a\gamma(a, x) - x^a e^{-x}, \tag{17}$$

$$\Gamma(a + 1, x) = a\Gamma(a, x) + x^a e^{-x}. \tag{18}$$

Special Values and Series. For integers $n \geq 1$,

$$\Gamma(n, x) = (n - 1)! e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}. \tag{19}$$

For $a > 0$ and $x \geq 0$,

$$\gamma(a, x) = x^a \sum_{n=0}^{\infty} \frac{(-x)^n}{n!(a + n)}. \tag{20}$$

The error function is related by

$$\gamma\left(\frac{1}{2}, x\right) = \sqrt{\pi}\ \mathrm{erf}(\sqrt{x}). \tag{21}$$

Complementarity and Regularization. Splitting Euler's integral into x gives the following:

$$\gamma(a, x) + \Gamma(a, x) = \Gamma(a). \tag{22}$$

Define

$$P(a, x) = \frac{\gamma(a, x)}{\Gamma(a)}, \qquad Q(a, x) = \frac{\Gamma(a, x)}{\Gamma(a)}. \tag{23}$$

Therefore,

$$P(a, x) + Q(a, x) = 1. \tag{24}$$

For $a > 0$ and $x \geq 0$, both functions lie between zero and one.

The Reciprocal Gamma Function

The reciprocal $1/\Gamma(z)$ is entire. Its product representation is

$$\frac{1}{\Gamma(z)} = z e^{\gamma_{\mathrm{E}} z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n},\ z \in \mathbb{C} \setminus \mathbb{Z}_0^-. \tag{25}$$

3|Certain Applications

Evaluating Impossible Integrals

Meaning of the Terminology. Here, "impossible integrals" means integrals not conveniently evaluated using elementary functions. It does not mean that the integrals are mathematically undefined or unevaluable.

A General Gamma Integral. For $\mu > 0$, $p > 0$, and $a > 0$, substitution of $t = \mu x^p$ gives

$$\int_0^\infty x^{a-1} e^{-\mu x^p}\, dx = \frac{1}{p}\mu^{-a/p}\Gamma\left(\frac{a}{p}\right). \tag{26}$$

Examples include

$$\int_0^\infty e^{-x^4}\, dx = \frac{1}{4}\Gamma\left(\frac{1}{4}\right), \tag{27}$$

$$\int_0^\infty x^2\, e^{-\beta x^2}\, dx = \frac{\sqrt{\pi}}{4\beta^{3/2}}, \qquad \beta > 0. \tag{28}$$

For $b > 0$, the finite-range version is

$$\int_0^b x^{a-1}\, e^{-\mu x^p}\, dx = \frac{1}{p}\mu^{-a/p}\gamma\left(\frac{a}{p}, \mu b^p\right). \tag{29}$$

Beta-Type Integrals. For $0 < a < b$,

$$\int_0^\infty \frac{x^{a-1}}{(1+x)^b}\, dx = \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)}. \tag{30}$$

Statistics and Physical Problems

Coulomb Wave Functions. (see [13, 14])

A standard representation of the regular Coulomb function is

$$F_\ell(\eta, \rho) = C_\ell(\eta)\rho^{\ell+1}\, e^{-i\rho}\ {}_1F_1(\ell + 1 - i\eta; 2\ell + 2; 2i\rho), \tag{31}$$

where

$$C_\ell(\eta) = \frac{2^\ell\, e^{-\pi\eta/2}\|\Gamma(\ell + 1 + i\eta)\|}{\Gamma(2\ell + 2)}. \tag{32}$$

Here ℓ is a nonnegative integer, η is real, and $\rho > 0$.

The Coulomb phase shift is

$$\sigma_\ell(\eta) = \mathrm{ph}\,\Gamma(\ell + 1 + i\eta), \tag{33}$$

with phase chosen continuously from zero at $\eta = 0$ [3, Sec. 33.2].

Clarify that C_ℓ is the conventional normalization factor for the Coulomb solution, not an assertion that a continuum scattering wave is square-integrable.

Canonical Energy Integrals. (see [15])

Suppose a model has density of states

$$g(E) = AE^{a-1}, \qquad E > 0, \quad a > 0. \tag{34}$$

For inverse temperature $\beta = (k_{\mathrm{B}}T)^{-1}$,

$$Z(\beta) = \int_0^\infty g(E)\, e^{-\beta E}\, dE = A\frac{\Gamma(a)}{\beta^a}. \tag{35}$$

If A is independent of β, then

$$\langle E\rangle = -\frac{\partial \log Z}{\partial \beta} = \frac{a}{\beta}. \tag{36}$$

This is a model-dependent result: it applies to the stated power-law density of states, not to every physical system.

Probability Theory

The Gamma Distribution. (see [14])

Using shape $a > 0$ and rate $\lambda > 0$, define

$$f_X(x) = \frac{\lambda^a}{\Gamma(a)} x^{a-1}\, e^{-\lambda x}, \qquad x > 0. \tag{37}$$

The rate convention must be distinguished from the scale convention, where the scale is $1/\lambda$.

Normalization follows directly from Euler's integral.

Moments and Tail Probabilities. For real $r > -a$,

$$\mathbb{E}[X^r] = \lambda^{-r} \frac{\Gamma(a+r)}{\Gamma(a)}. \tag{38}$$

Hence,

$$\mathbb{E}[X] = \frac{a}{\lambda}, \qquad \mathrm{Var}(X) = \frac{a}{\lambda^2}. \tag{39}$$

The cumulative and survival probabilities are

$$\Pr(X \le x) = P(a, \lambda x), \qquad \Pr(X > x) = Q(a, \lambda x). \tag{40}$$

Related Distributions. (see [13])

The exponential distribution corresponds to $a = 1$. For an integer shape, the gamma distribution is also called an Erlang distribution. A chi-squared distribution with ν degrees of freedom has shape $\nu/2$ and rate $1/2$.

Fractional Calculus

This part explains why the gamma function appears naturally when integration and differentiation are extended to noninteger orders (see [6, 7, 16, 17]).

From Repeated Integration to Fractional Integration. For a positive integer n, repeated integration gives

$$I_{0+}^n f(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)\, dt. \tag{41}$$

Replacing $(n-1)!$ by $\Gamma(n)$ motivates the following extension.

Definition 1. For $\alpha > 0$, the Riemann–Liouville fractional integral is

$$I_{0+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \tag{42}$$

whenever the integral exists.

For $\mu > -1$, $\alpha > 0$, and $x > 0$,

$$I_{0+}^{\alpha} x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} x^{\mu+\alpha}. \tag{43}$$

Derive this formula by setting $t = xu$ and applying Equation (7).

For example,

$$I_{0+}^{1/2} 1 = \frac{2\sqrt{x}}{\sqrt{\pi}}. \tag{44}$$

The Semigroup Property. Under suitable integrability assumptions,

$$I_{0+}^{\alpha} I_{0+}^{\beta} f = I_{0+}^{\alpha+\beta} f, \qquad \alpha, \beta > 0. \tag{45}$$

The proof uses an interchange of integrals and the beta identity. The gamma normalization is essential to obtaining the correct composition law.

Riemann–Liouville and Caputo Derivatives. Let $n - 1 < \alpha < n$, where n is a positive integer.

The Riemann–Liouville derivative is

$$D_{0+}^{\alpha} f(x) = \frac{d^n}{dx^n} I_{0+}^{n-\alpha} f(x). \tag{46}$$

The Caputo derivative is

$${}^{\mathrm{C}}D_{0+}^{\alpha} f(x) = I_{0+}^{n-\alpha} f^{(n)}(x). \tag{47}$$

Assume sufficient regularity for the stated operations; for example, the Caputo construction can be introduced for functions whose $(n-1)$st derivative is absolutely continuous on the interval under consideration.

Why the Definitions Are Not Interchangeable. For $0 < \alpha < 1$,

$$D_{0+}^{\alpha} 1 = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}, \qquad {}^{\mathrm{C}}D_{0+}^{\alpha} 1 = 0. \tag{48}$$

This example shows that fractional derivatives are not uniquely specified by their order alone. The operator definition and lower terminal must also be stated.

Mittag–Leffler Functions and Fractional Relaxation. (see [5, 6])

The Mittag–Leffler Function of two parameters define by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha > 0. \tag{49}$$

Write $E_{\alpha}(z) = E_{\alpha,1}(z)$. The identity $E_1(z) = e^z$ connects this family to the ordinary exponential. Now, we consider For $0 < \alpha < 1$ and $\lambda > 0$, the Caputo problem

$${}^{\mathrm{C}}D_{0+}^{\alpha} y(t) = -\lambda y(t), \qquad y(0) = y_0, \tag{50}$$

has the standard solution

$$y(t) = y_0 E_{\alpha}(-\lambda t^{\alpha}). \tag{51}$$

This example links gamma-function coefficients to the solution of a fractional differential equation (see [16, 17]).

4|Conclusion

The gamma function connects factorial interpolation, special-function identities, probability normalization, and noninteger-order operators. Its incomplete forms describe truncated integrals and probability tails, while its values at complex arguments enter Coulomb-wave representations. The reason for this review is that the gamma function is both an important analytical object and a structural component of a wider mathematical framework.

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