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An Introductory Overview of The $k-$Analogue of Gamma Functions and Related Functions

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Abstract

The main objective of this work is to introduce various representations of the k-analogue of the Gamma function. We prove some basic properties of the k-Gamma functions. In addition, we establish some related functions, such as the k-Beta function, the k-Digamma and k-Polygamma functions, and the k-incomplete Gamma functions.

Keywords: Special functions, Gamma function, Factorial generalization, Beta functions.

1|Introduction

At the beginning of 18^{th} century, there was a race among the mathematicians to generalize the factorial function in such a way that it could be defined on entire complex plane. The problem was first considered by Daniel Bernoulli and Christian Goldbach in the 1720s and it was solved at the end of the same decade by Leonhard Euler in 1729 finishes this and produces a generalization in terms of an integral known as Euler's second integral. He gave further representations by integrals, and formulated interesting theorems on the properties of this function. The historical development of the Gamma function as an extension of the factorial is well-documented. For more information, see the works by Davis [29], Artin [11], Borwein and Corless[12], Conway [14], and Freeden and Martin [24]. There are several definitions of the Gamma function, including the Weierstrass definition,

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Euler's integral definition, and Gauss definition. Also, the Gamma function holds a key place across many scientific and mathematical fields because of its nature and commonness in depicting complicated events. In probability and statistics, where it supports basic distributions like the Gamma and beta distributions utilized to represent waiting periods, dependability, and variance within Bayesian structures, with direct uses in areas from weather forecasting to money management and machine dependability studies (see, e.g., [1, 10, 22, 21, 31]). In physics and engineering, formulas including the Gamma function often show up in quantum mechanics, signal handling, and the study of flowing matter, especially when calculating integrals, end behaviors, and change methods such as the Laplace and Fourier transforms [6, 23, 27]. Furthermore, the Gamma function is essential to special function theory including hypergeometric and Bessel functions which are vital instruments for solving straight differential equations and picturing how waves move and spread in practical sciences [2, 3, 8, 15, 27].

Moreover, the history of k-functions theory is relatively recent. It began in the early years of 21st century when Diaz and Pariguan introduced the generalized k-Gamma function and investigated its fundamental properties [18]. The k-Gamma function Γ_k which is a one parameter deformation of the classical Gamma function Γ such that $\Gamma_k \to \Gamma$ as $k \to 1$ [30].
The main objective of this work is to introduce various representations of the k-analogue of the Gamma function. We prove some basic properties of the k-Gamma functions. In addition, we establish some related functions, such as the k-Beta function, the k-Digamma and k-Polygamma functions, and the k-incomplete Gamma functions.

2|The $k-$Analogue of the Gamma Function

The k-Gamma function, denoted by $\Gamma_k(z)$, is a generalization of the classical Gamma function introduced by Diaz and Pariguan [18].

Definition 1. For $k > 0$, the $k-$Gamma function is given by :

$$\Gamma_k(z) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{z}{k}-1}}{(z)_{n,k}}, \quad z \in \mathbb{C}\backslash k\mathbb{Z}^-. \tag{1}$$

Where the Pochhammer′s k-symbol is defined as

$$(z)_{n,k} = z(z+k)(z+2k)\dots(z+(n-1)k)\ , \quad z \in \mathbb{C}, \quad k > 0, \quad n \in \mathbb{N}. \tag{2}$$

Theorem 1. *(see* [18]*). For $k > 0$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. The k-Gamma function is defined by the following integral:*

$$\Gamma_k(z) = \int_0^\infty t^{z-1}e^{-\frac{t^k}{k}}dt. \tag{3}$$

Proof: The following proof is provided according to [18]. Using the classical limit representation

$$e^{-x} = \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n,$$

with

$$x = \frac{t^k}{k},$$

we obtain

$$e^{-\frac{t^k}{k}} = \lim_{n\to\infty}\left(1-\frac{t^k}{nk}\right)^n.$$

Therefore,

$$\int_0^\infty t^{z-1}e^{-\frac{t^k}{k}}dt = \lim_{n\to\infty}\int_0^{(nk)^{1/k}}\left(1-\frac{t^k}{nk}\right)^n t^{z-1}dt, \tag{4}$$

by writing

$$\Psi_{n,i}(z) = \int_0^{(nk)^{1/k}}\left(1-\frac{t^k}{nk}\right)^i t^{z-1}dt, \quad i = 0,1,2,\dots,n. \tag{5}$$

The following recursive formula is proven using integration by parts

$$\begin{aligned}\Psi_{n,i}(z) =& \int_0^{(nk)^{1/k}} \left(1-\frac{t^k}{nk}\right)^i t^{z-1}dt \\ =& \left[\frac{1}{z}\left(1-\frac{t^k}{nk}\right)^i t^z\right]_0^{(nk)^{1/k}} + \frac{i}{z}\int_0^{(nk)^{1/k}} \left(1-\frac{t^k}{nk}\right)^{i-1} \frac{t^{k-1}}{n} t^z dt \\ =& \frac{i}{nz}\int_0^{(nk)^{1/k}} \left(1-\frac{t^k}{nk}\right)^{i-1} t^{z+k-1}dt \\ =& \frac{i}{nz}\Psi_{n,i-1}(z+k),\end{aligned}$$

when $i=0$, we have

$$\Psi_{n,0}(z) = \int_0^{(nk)^{1/k}} t^{z-1}dt = \frac{(nk)^{z/k}}{z}.$$

When $i=1,2,\ldots,n$, we get

$$\begin{aligned}\Psi_{n,1}(z) =& \frac{1}{nz}\Psi_{n,0}(z+k) = \frac{1}{nz}\int_0^{(nk)^{1/k}} t^{z+k-1}dt = \frac{1}{nz}\frac{(nk)^{\frac{z+k}{k}}}{z+k} \\ \Psi_{n,2}(z) =& \frac{2}{nz}\Psi_{n,1}(z+k) = \frac{2}{nz}\int_0^{(nk)^{1/k}} \left(1-\frac{t^k}{nk}\right) t^{z+k-1}dt = \frac{1}{nz}\frac{2}{n(z+k)}\frac{(nk)^{\frac{z+2k}{k}}}{z+2k} \\ \Psi_{n,n}(z) =& \frac{n}{nz}\Psi_{n,n-1}(z+k) \\ =& \frac{1}{nz}\frac{2}{n(z+k)}\cdots\frac{n}{n(z+(n-1)k)}\frac{(nk)^{\frac{z+nk}{k}}}{z+nk} \\ =& \frac{n!k^n(nk)^{\frac{z}{k}-1}}{(z)_{n,k}\left(1+\frac{z}{nk}\right)}.\end{aligned}$$

By virtue of (1), we obtain

$$\begin{aligned}\lim_{n\to\infty}\Psi_{n,n}(z) &= \lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{z}{k}-1}}{(z)_{n,k}\left(1+\frac{z}{nk}\right)} \\ &= \lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{z}{k}-1}}{(z)_{n,k}} = \Gamma_k(z),\end{aligned} \tag{6}$$

which gives the desired formula in (3). □

Some Properties of the k-Gamma Function

Here, we present some fundamental properties of the k-Gamma function.

Theorem 2. *(see*[18]*). For $k\in\mathbb{R}^+$ and $z\in\mathbb{C}\backslash kZ^-$, the k-Gamma function satisfies the following relation:*

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right). \tag{7}$$

Proof: To show that, putting $u = \frac{t^k}{k}$ in (3), then $du = t^{k-1}dt$

$$\begin{aligned}\Gamma_k(z) &= \int_0^\infty t^{z-1}e^{-\frac{t^k}{k}}dt = \int_0^\infty t^{z-k}e^{-\frac{t^k}{k}}t^{k-1}dt \\ &= \int_0^\infty \left(t^k\right)^{\frac{z}{k}-1}e^{-\frac{t^k}{k}}t^{k-1}dt = k^{\frac{z}{k}-1}\int_0^\infty (u)^{\frac{z}{k}-1}e^{-u}du \\ &= k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right).\end{aligned}$$

□

Remark 1. We can rewrite the relation (7) in the form

$$\Gamma_k(wk) = k^{w-1}\Gamma(w); \quad k > 0, w \in \mathbb{C}\backslash k\mathbb{Z}^-. \tag{8}$$

Theorem 3. *(see* [18]*). For $k \in \mathbb{R}^+$ and $z \in \mathbb{C}\backslash kZ^-$. The k-Gamma function satisfies the following properties:*

$$\Gamma_k(k) = 1, \tag{9}$$

$$\Gamma_k(z+k) = z\Gamma_k(z), \tag{10}$$

$$(z)_{n,k} = \frac{\Gamma_k(z+nk)}{\Gamma_k(z)}, \tag{11}$$

$$\frac{1}{\Gamma_k(z)} = zk^{-\frac{z}{k}}e^{\frac{z}{k}\gamma}\prod_{n=1}^{\infty}\left(\left(1+\frac{z}{nk}\right)e^{-\frac{z}{nk}}\right), \tag{12}$$

where

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln(n)\right) \approx 0.577216, \tag{13}$$

Proof: To prove property (9) using (7) with $\Gamma(1) = 1$, we get the desired result.

By invoking of (7), we have,

$$\Gamma_k(z+k) = k^{\frac{z+k}{k}-1}\Gamma\left(\frac{z+k}{k}\right).$$

From the above, we can write,

$$k^{\frac{z+k}{k}-1}\Gamma\left(\frac{z+k}{k}\right) = k^{\frac{z}{k}}\frac{z}{k}\Gamma\left(\frac{z}{k}\right) = zk^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right) = z\Gamma_k(z),$$

which yields the coveted property (10).

Now, to demonstrate property (11). By using the relation (10), we get

$$\Gamma_k(z+nk) = z(z+k)(z+2k)\ldots(z+(n-1)k)\Gamma_k(z),$$

from (2), it follows that

$$\Gamma_k(z+nk) = (z)_{n,k}\Gamma_k(z).$$

Thus, the property is as described in (11).

$$(z)_{n,k} = \frac{\Gamma_k(z+nk)}{\Gamma_k(z)}.$$

To show property (12), from (**??**) and (7), we have that

$$\begin{aligned}\Gamma_k(z) &= k^{\frac{z}{k}-1}\Gamma(\frac{z}{k}) \\ &= k^{\frac{z}{k}-1}\frac{e^{-\gamma\frac{z}{k}}}{\frac{z}{k}}\prod_{n=1}^{\infty}\left(1+\frac{z}{nk}\right)^{-1}e^{\frac{z}{nk}} \\ &= \frac{k^{\frac{z}{k}}}{z}e^{-\gamma\frac{z}{k}}\prod_{n=1}^{\infty}\left(1+\frac{z}{nk}\right)^{-1}e^{\frac{z}{nk}},\end{aligned}$$

thus,

$$\frac{1}{\Gamma_k(z)} = \frac{1}{k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)} = zk^{-\frac{z}{k}}e^{\frac{z}{k}\gamma}\prod_{n=1}^{\infty}\left(\left(1+\frac{z}{nk}\right)e^{-\frac{z}{nk}}\right).$$

□

Theorem 4. *(see* [18]*). For $k \in \mathbb{R}^+$ and $z \in \mathbb{C}\backslash kZ^-$. The k-Gamma function satisfies the following relation:*

$$\Gamma_k(z)\Gamma_k(k-z) = \frac{\pi}{k\sin\left(\frac{\pi z}{k}\right)}. \tag{14}$$

Proof: To prove that, from (7), we consider

$$\begin{aligned}\Gamma_k(z)\Gamma_k(k-z) =& k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)k^{\frac{k-z}{k}-1}\Gamma\left(\frac{k-z}{k}\right) \\ =& k^{-1}\Gamma\left(\frac{z}{k}\right)\Gamma\left(\frac{k-z}{k}\right).\end{aligned}$$

Using the relation

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin\pi z}, \qquad z \in \mathbb{C}\backslash\mathbb{Z}^-,$$

we get

$$k^{-1}\Gamma\left(\frac{z}{k}\right)\Gamma\left(\frac{k-z}{k}\right) = \frac{\pi}{k\sin\left(\frac{\pi z}{k}\right)}.$$

Thus, we get the required result in (14). □

Remark 2. (see [18]). For $k > 0$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. The k-Gamma function satisfies the following relation:

$$\Gamma_k(z) = a^{\frac{z}{k}}\int_0^{\infty} t^{z-1}e^{-\frac{t^k}{k}a}dt; \quad a > 0. \tag{15}$$

Proof: To demonstrate this result, let $t = a^{\frac{1}{k}}s$. Thus, $t^k = as^k$. From (3), we get :

$$\begin{aligned}\Gamma_k(z) =& \int_0^{\infty} t^{z-1}e^{-\frac{t^k}{k}}dt \\ =& a^{\frac{1}{k}}\int_0^{\infty}\left(a^{\frac{1}{k}}s\right)^{z-1}e^{-\frac{s^k}{k}a}ds \\ =& a^{\frac{z}{k}}\int_0^{\infty} s^{z-1}e^{-\frac{s^k}{k}a}ds.\end{aligned}$$

This implies that

$$\Gamma_k(z) = a^{\frac{z}{k}}\int_0^{\infty} t^{z-1}e^{-\frac{t^k}{k}a}dt.$$

□

Theorem 5. *(see* [18]*). Let $z \in \mathbb{C}\backslash kZ^-$ and $k, s \in \mathbb{R}^+$, the following identity holds:*

$$(z)_{n,s} = \left(\frac{s}{k}\right)^n \left(\frac{kz}{s}\right)_{n,k}. \tag{16}$$

Proof: From definition of the Pochhammer k-symbol, it follows that

$$\begin{aligned}(z)_{n,s} &= z(z+s)(z+2s)\dots(z+(n-1)s)\\ &= \frac{s}{k}\frac{kz}{s}\left(\frac{s}{k}\frac{kz}{s}+s\right)\left(\frac{s}{k}\frac{kz}{s}+2s\right)\dots\left(\frac{s}{k}\frac{kz}{s}+(n-1)s\right)\\ &= \left(\frac{s}{k}\right)^n \frac{kz}{s}\left(\frac{kz}{s}+k\right)\left(\frac{kz}{s}+2k\right)\dots\left(\frac{kz}{s}+(n-1)k\right)\\ &= \left(\frac{s}{k}\right)^n \left(\frac{kz}{s}\right)_{n,k}.\end{aligned}$$

Thus, this Theorem is proved. □

Theorem 6. *(see* [18]*). Let $z \in \mathbb{C}\backslash kZ^-$ and $k, s \in \mathbb{R}^+$, the following identity holds:*

$$\Gamma_s(z) = \left(\frac{s}{k}\right)^{\frac{z}{s}-1} \Gamma_k\left(\frac{kz}{s}\right). \tag{17}$$

Proof: Applying the relation (1), we verify that

$$\begin{aligned}\Gamma_s(z) &= \lim_{n\to\infty} \frac{n!s^n(ns)^{\frac{z}{s}-1}}{(z)_{n,s}}\\ &= \lim_{n\to\infty} \frac{n!k^n\left(\frac{s}{k}\right)^n\left(\frac{s}{k}\right)^{\frac{z}{s}-1}(nk)^{\left(\frac{kz}{s}\right)/k-1}}{\left(\frac{s}{k}\right)^n\left(\frac{kz}{s}\right)_{n,k}}\\ &= \left(\frac{s}{k}\right)^{\frac{z}{s}-1}\Gamma_k\left(\frac{kz}{s}\right).\end{aligned}$$

Therefore, the Theorem is established. □

Theorem 7. *(see* [16]*). For $\mathrm{Re}(z) > 0, \mathrm{Re}(w) > 0; 1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. the following inequality holds:*

$$\left|\Gamma_k\left(\frac{z}{p}+\frac{w}{q}\right)\right| \le \left|\Gamma_k(z)\right|^{\frac{1}{p}}\left|\Gamma_k(w)\right|^{\frac{1}{q}}. \tag{18}$$

Proof: Using relation (3), we have

$$\begin{aligned}\Gamma_k\left(\frac{z}{p}+\frac{w}{q}\right) &= \int_0^\infty t^{\left(\frac{z}{p}+\frac{w}{q}\right)-1}e^{-\frac{t^k}{k}}\,dt\\ &= \int_0^\infty t^{\left(\frac{z}{p}+\frac{w}{q}\right)-\left(\frac{1}{p}+\frac{1}{q}\right)}e^{-\frac{t^k}{k}}\,dt\\ &= \int_0^\infty t^{\left(\frac{z-1}{p}+\frac{w-1}{q}\right)}e^{-\left(\frac{1}{p}+\frac{1}{q}\right)\frac{t^k}{k}}dt\\ &= \int_0^\infty t^{\frac{z-1}{p}}e^{-\frac{t^k}{pk}}\;t^{\frac{w-1}{q}}e^{-\frac{t^k}{qk}}\,dt,\end{aligned}$$

and by employing Lemma in [20], we obtain

$$\begin{aligned}
&\left| \int_0^\infty t^{\frac{z-1}{p}} e^{-\frac{t^k}{pk}}\ t^{\frac{w-1}{q}} e^{-\frac{t^k}{qk}}\ dt \right| \\
&\leq \left[\int_0^\infty \left| t^{\frac{z-1}{p}} e^{-\frac{t^k}{pk}} \right|^p dt \right]^{\frac{1}{p}} \left[\int_0^\infty \left| t^{\frac{w-1}{q}} e^{-\frac{t^k}{qk}} \right|^q dt \right]^{\frac{1}{q}} \\
&= \left[\int_0^\infty \left| t^{z-1} e^{-\frac{t^k}{k}} \right| dt \right]^{\frac{1}{p}} \left[\int_0^\infty \left| t^{w-1} e^{-\frac{t^k}{k}} \right| dt \right]^{\frac{1}{q}} \\
&= \left| \Gamma_k(z) \right|^{\frac{1}{p}} \left| \Gamma_k(w) \right|^{\frac{1}{q}}.
\end{aligned}$$

Hence,

$$\left| \Gamma_k \left(\frac{z}{p} + \frac{w}{q} \right) \right| \leq \left| \Gamma_k(z) \right|^{\frac{1}{p}} \left| \Gamma_k(w) \right|^{\frac{1}{q}}.$$

□

3|Associated Functions of the k-Gamma Function and Their Properties

The k-Beta function. The k-Beta function can be defined as follows [18]

$$\mathrm{B}_k(z_1, z_2) = \frac{1}{k} \int_0^1 t^{\frac{z_1}{k}-1} (1-t)^{\frac{z_2}{k}-1}\ dt, \quad k > 0,\ \mathrm{Re}(z_1) > 0, \mathrm{Re}(z_2) > 0. \tag{19}$$

Remark 3. From (**??**) and (19), we deduce the following formula

$$\mathrm{B}_k(z_1, z_2) = \frac{1}{k} \mathrm{B}(\frac{z_1}{k}, \frac{z_2}{k}). \tag{20}$$

Thus, we get

$$\mathrm{B}_k(z_1, z_2) = \frac{\Gamma_k(z_1)\Gamma_k(z_2)}{\Gamma_k(z_1 + z_2)}, \quad k > 0,\ \mathrm{Re}(z_1) > 0, \mathrm{Re}(z_2) > 0. \tag{21}$$

Let $z_1 = z_2$ in (21) and utilize (20), the k-analogue Legendre duplication formula follows

$$\Gamma_k(2z_1) = 2^{2\frac{z_1}{k} - \frac{1}{2}} k^{\frac{1}{2}} (2\pi)^{-\frac{1}{2}} \Gamma_k(z_1)\Gamma_k(z_1 + \frac{k}{2}), \quad k > 0,\ \mathrm{Re}(z_1) > 0, \mathrm{Re}(z_2) > 0. \tag{22}$$

Theorem 8. *(see* [18]*). For $k \in \mathbb{R}^+$, the k-Beta function satisfies the following properties*

$$\mathrm{B}_k(z_1, z_2) = \int_0^\infty t^{z_1 - 1} (1 + t^k)^{-\frac{z_1 + z_2}{k}}\ dt, \tag{23}$$

$$\mathrm{B}_k(z_1, z_2) = \frac{z_1 + z_2}{z_1 z_2} \prod_{n=0}^{\infty} \frac{nk(nk + z_1 + z_2)}{(nk + z_1)(nk + z_2)}. \tag{24}$$

Proof: To establish property (23), from (20), we find

$$\mathrm{B}_k(z_1, z_2) = \frac{1}{k} \mathrm{B}(\frac{z_1}{k}, \frac{z_2}{k}) = \frac{1}{k} \int_0^1 u^{\frac{z_1}{k}-1} (1-u)^{\frac{z_2}{k}-1} du.$$

Let $u = \frac{1}{1+t^k}$, we obtain

$$1 - u = \frac{t^k}{1 + t^k}, \qquad du = -kt^{k-1}(1 + t^k)^{-2} dt.$$

Moreover, when $u \to 0$ then $t \to \infty$, and when $u \to 1$ then $t \to 0$.

$$\begin{aligned}\frac{1}{k}\int_0^1 u^{\frac{z_1}{k}-1}(1-u)^{\frac{z_2}{k}-1}du &= -\frac{1}{k}\int_\infty^0 \left(\frac{1}{1+t^k}\right)^{\frac{z_1}{k}-1}\left(1-\frac{1}{1+t^k}\right)^{\frac{z_2}{k}-1} kt^{k-1}(1+t^k)^{-2}dt\\ &= \int_0^\infty \left(\frac{1}{1+t^k}\right)^{\frac{z_1}{k}+1}\left(\frac{1}{1+t^k}\right)^{\frac{z_2}{k}-1} t^{z_2-k}t^{k-1}dt\\ &= \int_0^\infty \left(1+t^k\right)^{-\frac{z_1}{k}-1}\left(1+t^k\right)^{-\frac{z_2}{k}+1} t^{z_2-1}dt\\ &= \int_0^\infty \left(1+t^k\right)^{-\frac{z_1+z_2}{k}} t^{z_2-1}dt.\end{aligned}$$

Therefore, we obtain the required formula.

To prove the property (24), from the relation (21), we use

$$\mathrm{B}_k(z_1,z_2) = \frac{\Gamma_k(z_1)\Gamma_k(z_2)}{\Gamma_k(z_1+z_2)},$$

Property (12), gives

$$\begin{aligned}\Gamma_k(z_1)\Gamma_k(z_2) &= \left(\frac{k^{\frac{z_1}{k}}}{z_1}e^{-\gamma\frac{z_1}{k}}\prod_{n=1}^\infty\left(1+\frac{z_1}{nk}\right)^{-1}e^{\frac{z_1}{nk}}\right)\left(\frac{k^{\frac{z_2}{k}}}{z_2}e^{-\gamma\frac{z_2}{k}}\prod_{n=1}^\infty\left(1+\frac{z_2}{nk}\right)^{-1}e^{\frac{z_2}{nk}}\right)\\ &= \frac{k^{\frac{z_1+z_2}{k}}}{z_1z_2}e^{-\gamma\frac{z_1+z_2}{k}}\prod_{n=1}^\infty\frac{1}{\left(1+\frac{z_1}{nk}\right)\left(1+\frac{z_2}{nk}\right)}e^{\frac{z_1+z_2}{nk}},\end{aligned}$$

and

$$\frac{1}{\Gamma_k(z_1+z_2)} = (z_1+z_2)k^{-\frac{z_1+z_2}{k}}e^{\frac{z_1+z_2}{k}\gamma}\prod_{n=1}^\infty\left(\left(1+\frac{z_1+z_2}{nk}\right)e^{-\frac{z_1+z_2}{nk}}\right),$$

thus,

$$\begin{aligned}\mathrm{B}_k(z_1,z_2 &= \left(\frac{k^{\frac{z_1+z_2}{k}}}{z_1z_2}e^{-\frac{z_1+z_2}{k}\gamma}\prod_{n=1}^\infty\frac{1}{\left(1+\frac{z_1}{nk}\right)\left(1+\frac{z_2}{nk}\right)}e^{\frac{z_1+z_2}{nk}}\right)\\ &\times\left((z_1+z_2)k^{-\frac{z_1+z_2}{k}}e^{\frac{z_1+z_2}{k}\gamma}\prod_{n=1}^\infty\left(1+\frac{z_1+z_2}{nk}\right)e^{-\frac{z_1+z_2}{nk}}\right)\\ &= \frac{z_1+z_2}{z_1z_2}\prod_{n=1}^\infty\frac{1}{\left(1+\frac{z_1}{nk}\right)\left(1+\frac{z_2}{nk}\right)}\left(1+\frac{z_1+z_2}{nk}\right)\\ &= \frac{z_1+z_2}{z_1z_2}\prod_{n=0}^\infty\frac{nk(nk+z_1+z_2)}{(nk+z_1)(nk+z_2)}.\end{aligned}$$

This gives the desired formula (24). □

Theorem 9. *(see* [25, 26, 28]*). For $k \in \mathbb{R}^+$ and $z_1, z_2 \in \mathbb{C}$ with $\mathrm{Re}(z_1, z_2) > 0$, the k-Beta function satisfies the following properties*

$$\mathrm{B}_k(z_1+k,z_2) = \frac{z_1}{z_1+z_2}\mathrm{B}_k(z_1,z_2), \tag{25}$$

$$\mathrm{B}_k(z_1k,z_2k) = \frac{1}{k}\mathrm{B}(z_1,z_2), \tag{26}$$

$$\mathrm{B}_k(z_1,k) = \frac{1}{z_1}, \tag{27}$$

Proof: Using definition (21), we demonstrate that

$$\mathrm{B}_k(z_1+k,z_2)=\frac{\Gamma_k(z_1+k)\Gamma_k(z_2)}{\Gamma_k(z_1+k+z_2)}.$$

By using the property (10), we get

$$\begin{aligned}\frac{\Gamma_k(z_1+k)\Gamma_k(z_2)}{\Gamma_k(z_1+z_2+k)}&=\frac{z_1\Gamma_k(z_1)\Gamma_k(z_2)}{(z_1+z_2)\Gamma_k(z_1+z_2)}\\&=\frac{z_1}{z_1+z_2}\frac{\Gamma_k(z_1)\Gamma_k(z_2)}{\Gamma_k(z_1+z_2)}\\&=\frac{z_1}{z_1+z_2}\mathrm{B}_k(z_1,z_2),\end{aligned}$$

which gives the desired formula (25).

To prove the property (26), we utilize the property (20) as

$$\mathrm{B}_k(z_1k,z_2k)=\frac{1}{k}\mathrm{B}(\frac{z_1k}{k},\frac{z_2k}{k})=\frac{1}{k}\mathrm{B}(z_1,z_2)$$

which yields the required property (26).

To show the property (27), from the definition (21), we obtain

$$\mathrm{B}_k(z_1,k)=\frac{\Gamma_k(z_1)\Gamma_k(k)}{\Gamma_k(z_1+k)},$$

using properties (9) and (10), we obtain the following result

$$\mathrm{B}_k(z_1,k)=\frac{\Gamma_k(z_1)\Gamma_k(k)}{\Gamma_k(z_1+k)}=\frac{\Gamma_k(z_1)\times 1}{z_1\Gamma_k(z_1)}=\frac{1}{z_1}.$$

□

Remark 4. For $k=1$ in Theorem8 and Theorem9, we obtain the results of the classical Beta function in [6, 9, 11].

The $k-$Digamma and $k-$Polygamma Functions. For $\mathrm{Re}(z),k>0$, we define the k-Digamma function and the $k-$Polygamma function respectively by, (see [7]).

$$\psi_k(z)=\frac{d}{dz}\log\Gamma_k(z)=\frac{\Gamma_k'(z)}{\Gamma_k(z)},\tag{28}$$

and

$$\psi_k^{(i)}(z)=\frac{d^i}{dz^i}\log\Gamma_k(z),\tag{29}$$

where

$$\log\Gamma_k(z)=\log\Gamma_k(z+k)-\log z.\tag{30}$$

From (28) and (30), we can write

$$\psi_k(z+k)=\psi_k(z)+\frac{1}{z}.\tag{31}$$

The relation (31) is sometimes called the functional equation of $k-$Digamma function.

Taking the logarithmic derivative of the inverse of the k-analogue Weierstrass form given in Theorem3, we have

$$\Gamma_k(z)=z^{-1}k^{\frac{z}{k}}e^{-\frac{z}{k}\gamma}\prod_{n=1}^{\infty}\left(\frac{nk}{z+nk}\right)e^{\frac{z}{nk}}.\tag{32}$$

To justify differentiation term-by-term in the complex setting, define

$$f_n(z) = \frac{z}{nk} - \log\left(1 + \frac{z}{nk}\right), \quad z \in \Omega = \mathbb{C} \setminus \{-k, -2k, \dots\},$$

where a branch of the logarithm is fixed on Ω. Then each f_n is holomorphic on Ω, and

$$f_n'(z) = \frac{1}{nk} - \frac{1}{nk+z} = \frac{z}{nk(nk+z)}.$$

Let $K \subset \Omega$ be compact. Since K is compact and disjoint from $\{-k, -2k, \dots\}$, there exists a constant $c > 0$ such that

$$|nk + z| \geq cn, \quad \forall z \in K,\ n \geq 1.$$

Moreover, since K is compact, there exists $M > 0$ such that

$$|z| \leq M, \qquad z \in K.$$

Hence, for $z \in K$,

$$|f_n'(z)| = \left|\frac{z}{nk(nk+z)}\right| \leq \frac{M}{kcn^2}.$$

Since

$$\sum_{n=1}^{\infty} \frac{M}{kcn^2},$$

converges, the comparison convergence test implies that

$$\sum_{n=1}^{\infty} f_n'(z),$$

converges uniformly on K.

Next, at $z = 1$,

$$f_n(1) = \frac{1}{nk} - \log\left(1 + \frac{1}{nk}\right) = \frac{1}{2n^2k^2} + O\left(\frac{1}{n^3}\right),$$

so the series

$$\sum_{n=1}^{\infty} f_n(1),$$

converges.

Therefore, by the theorem on locally uniform convergence of derivatives of holomorphic series, the series

$$\sum_{n=1}^{\infty} f_n(z),$$

converges locally uniformly on Ω to a holomorphic function, and

$$\frac{d}{dz}\sum_{n=1}^{\infty} f_n(z) = \sum_{n=1}^{\infty} f_n'(z).$$

Consequently,

$$\frac{d}{dz}\sum_{n=1}^{\infty}\left[\frac{z}{nk}-\log\left(1+\frac{z}{nk}\right)\right]=\sum_{n=1}^{\infty}\left(\frac{1}{nk}-\frac{1}{nk+z}\right).$$

Thus

$$\psi_k(z)=-\frac{1}{z}+\frac{1}{k}\log k-\frac{\gamma}{k}+\sum_{n=1}^{\infty}\left(\frac{1}{nk}-\frac{1}{z+nk}\right). \tag{33}$$

By shifting the index of the series (33), we obtain

$$\psi_k(z)=\frac{1}{k}\log k-\frac{\gamma}{k}+\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)k}-\frac{1}{nk+z}\right). \tag{34}$$

Euler-Mascheroni constant and $k-$Digamma function. Now, we represent the Euler-Mascheroni constant in the form $k-$Digamma function for different values of k (cf. [7]).

Substituting $k=4$ and $z=1$ in (33) , we see that

$$\psi_4(1)=-1+\frac{1}{4}\log 4-\frac{\gamma}{4}+\sum_{n=1}^{\infty}\left(\frac{1}{4n}-\frac{1}{4n+1}\right). \tag{35}$$

The series in (35) is not hard to sum. First, we observe that this series can be written as

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{4n}-\frac{1}{4n+1}\right)&=\sum_{n=1}^{\infty}\int_0^1\left(u^{4n-1}-u^{4n}\right)du=\int_0^1\sum_{n=1}^{\infty}\left(u^{4n-1}-u^{4n}\right)du\\&=\int_0^1\left(u^3\sum_{n=1}^{\infty}\left(u^4\right)^{n-1}-u^4\sum_{n=1}^{\infty}\left(u^4\right)^{n-1}\right)du\\&=\int_0^1\left(\frac{u^3}{1-u^4}-\frac{u^4}{1-u^4}\right)du\\&=\int_0^1\frac{u^3}{(1+u)\left(1+u^2\right)}du\\&=\int_0^1\left(1-\frac{1}{2(1+u)}-\frac{u+1}{2\left(1+u^2\right)}\right)du.\end{aligned}$$

Evaluating the integral, we get the sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n}-\frac{1}{4n+1}\right)=1-\frac{\pi}{8}-\frac{3}{4}\log(2). \tag{36}$$

Substituting (35) into (36), we get a nice representation of the Euler-Mascheroni constant in terms of 4-Digamma function

$$\gamma=-4\psi_4(1)-\frac{\pi}{2}-\log(2), \tag{37}$$

or

$$\pi+2\gamma=-8\psi_4(1)-2\log(2).$$

By invoking the relation (cf. [7])

$$\frac{\pi}{2}=\psi_2\left(\frac{3}{2}\right)-\psi_2\left(\frac{1}{2}\right)=2\left(\psi_4(3)-\psi_4(1)\right). \tag{38}$$

and (37), we get some more such representations

$$\begin{aligned}
\gamma &= -4\psi_4(3) + \frac{\pi}{2} - \log(2), \\
\gamma &= -4\psi_4(3) + \psi_2\left(\frac{3}{2}\right) - \psi_2\left(\frac{1}{2}\right) - \log(2), \\
\gamma &= -4\psi_4(3) + \psi_2\left(\frac{3}{2}\right) - \psi_2\left(\frac{1}{2}\right) + \frac{1}{4}\psi_{\frac{1}{2}}\left(\frac{3}{4}\right) - \frac{1}{4}\psi_{\frac{1}{2}}\left(\frac{1}{2}\right) - 1.
\end{aligned} \tag{39}$$

Subtracting (37) from (39), we get

$$\pi = 4\left[\psi_4(3) - \psi_4(1)\right].$$

The $k-$Incomplete Gamma Functions

The incomplete Gamma function is a generalized form of the Gamma function, used to evaluate the function over a partial range of integration. By analogy with the classical case, and following the definition of the $k-$Gamma function given in Díaz et al. [17, 16, 19] and other articles [4, 5], we define the $k-$incomplete Gamma function as follows:

Lower $k-$incomplete Gamma function:

$$\gamma_k(z,y) = \int_0^y t^{z-1}e^{-\frac{t^k}{k}}dt, \quad \mathrm{Re}(z) > 0,\ y,k \in \mathbb{R}^+. \tag{40}$$

Upper $k-$incomplete Gamma function:

$$\Gamma_k(z,y) = \int_y^\infty t^{z-1}e^{-\frac{t^k}{k}}dt, \quad \mathrm{Re}(z) > 0,\ y,k \in \mathbb{R}^+. \tag{41}$$

They satisfy the following formula:

$$\Gamma_k(z,y) + \gamma_k(z,y) = \Gamma_k(z). \tag{42}$$

Thus, the $k-$incomplete Gamma functions are defined exactly in analogy to the $k-$Gamma function, just as the classical incomplete Gamma functions are defined in analogy to the classical Gamma function.

Theorem 10. *For $k, y > 0$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, the lower $k-$incomplete Gamma function satisfies the following relation:*

$$\gamma_k(z,y) = k^{\frac{z}{k}-1}\gamma(\frac{z}{k}, \frac{y^k}{k}). \tag{43}$$

Proof: We consider

$$\gamma_k(z,y) = \int_0^y t^{z-1}e^{-\frac{t^k}{k}}dt. \tag{44}$$

Let $u = t^k/k$, then $du = t^{k-1}dt \Rightarrow dt = du/t^{k-1} = du/(ku)^{(k-1)/k}$, thus

$$\begin{aligned}
\gamma_k(z,y) &= \int_0^{y^k/k} (ku)^{\frac{z-1}{k}} e^{-u}\ \frac{du}{(ku)^{\frac{k-1}{k}}} \\
&= \int_0^{y^k/k} (ku)^{\frac{z}{k}-1} e^{-u}\ du \\
&= k^{\frac{z}{k}-1} \int_0^{y^k/k} u^{\frac{z}{k}-1} e^{-u}\ du,
\end{aligned}$$

by applying the lower incomplete Gamma function, we arrive at the result (43).

□

Theorem 11. *For $k, y > 0$ and $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, the upper $k-$incomplete Gamma function satisfies the following relation:*

$$\Gamma_k(z, y) = k^{\frac{z}{k}-1}\Gamma(\frac{z}{k}, \frac{y^k}{k}). \tag{45}$$

Proof: We observe that

$$\Gamma_k(z, y) = \int_y^\infty t^{z-1}e^{-\frac{t^k}{k}}dt. \tag{46}$$

Let $w = t^k/k$. Then $dw = t^{k-1}dt \Rightarrow dt = dw/t^{k-1} = dw/(kw)^{(k-1)/k}$, thus

$$\begin{aligned}\Gamma_k(z, y) &= \int_{y^k/k}^\infty (kw)^{\frac{z-1}{k}}e^{-w}\ \frac{dw}{(kw)^{\frac{k-1}{k}}} \\ &= \int_{y^k/k}^\infty (kw)^{\frac{z}{k}-1}e^{-w}\ dw \\ &= k^{\frac{z}{k}-1}\int_{y^k/k}^\infty w^{\frac{z}{k}-1}e^{-w}\ dw,\end{aligned}$$

using the upper incomplete Gamma function, we get the relation (45).

□

4|Series Representation of the k-Gamma Function

Theorem 12. *(see* [32]*). The k-Gamma function has the following representation*

$$\Gamma_k(z) = 2\pi\sum_{n=0}^\infty \frac{\left(-\frac{1}{k}\right)^n}{n!}\delta(\theta - i(v + nk)), \tag{47}$$

where $\delta(.)$ is the Dirac delta function given in Definition [32]*.*

Proof: Recall that

$$\Gamma_k(z) = \int_0^\infty t^{z-1}e^{-\frac{t^k}{k}}dt, \qquad \mathrm{Re}(z) > 0.$$

Applying the change of variables $t = e^x$ and $z = v + i\theta$, we get

$$\Gamma_k(z) = \int_{-\infty}^\infty e^{x(v+i\theta)}\exp\left(-\frac{e^{kx}}{k}\right)dx, \quad k \in R^+.$$

Now, we express the series form of the exponential function

$$\exp\left(-\frac{e^{kx}}{k}\right) = \sum_{n=0}^\infty \frac{\left(-\frac{e^{kx}}{k}\right)^n}{n!},$$

then collecting and expanding the exponential terms

$$\Gamma_k(z) = \int_{-\infty}^\infty e^{x(v+i\theta)}\sum_{n=0}^\infty \frac{\left(-\frac{e^{kx}}{k}\right)^n}{n!}dx, \quad k \in R^+.$$

We have

$$\Gamma_k(z) = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{k}\right)^n}{n!} \int_{-\infty}^{\infty} e^{ix\theta} e^{(v+nk)x}\, dx.$$

Invoking the Fourier transform $\mathcal{F}\left[e^{\alpha t};\theta\right] = 2\pi\delta(\theta - i\alpha)$ (cf. [32]), we have

$$\int_{-\infty}^{\infty} e^{i\theta x} e^{(v+nk)x} dx = \mathcal{F}\left[e^{(v+nk)x};\theta\right] = 2\pi\delta(\theta - i(v+nk)), \tag{48}$$

therefore, we arrive at

$$\Gamma_k(z) = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{k}\right)^n}{n!} \int_{-\infty}^{\infty} e^{i\theta x} e^{(v+nk)x} dx = 2\pi \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{k}\right)^n}{n!} \delta(\theta - i(v+nk)).$$

□

Corollary 0.1. *(see* [32]*). The k-Gamma function have the subsequent presentation:*

$$\Gamma_k(z) = 2\pi \sum_{n,r=0}^{\infty} \frac{\left(-\frac{1}{k}\right)^n}{n!} \frac{(-i(v+nk))^r}{r!} \delta^{(r)}(\theta). \tag{49}$$

Proof: By using basic identity $\delta(z+b) = \sum_{i=0}^{\infty} \delta^{(i)}(z) \frac{b^i}{i!}$, we get

$$\delta(\theta - i(v+nk)) = \sum_{r=0}^{\infty} \delta^{(r)}(\theta) \frac{(-i(v+nk))^r}{r!}.$$

Now, from (47), we have

$$\begin{aligned}\Gamma_k(z) =& 2\pi \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{k}\right)^n}{n!} \delta(\theta - i(v+nk)) \\ =& 2\pi \sum_{n,r=0}^{\infty} \frac{\left(-\frac{1}{k}\right)^n}{n!} \frac{(-i(v+nk))^r}{r!} \delta^{(r)}(\theta).\end{aligned}$$

□

Corollary 0.2. *(see* [32]*). The k-Gamma functions have the subsequent demonstration:*

$$\Gamma_k(z) = 2\pi \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{k}\right)^n}{n!} \delta(z+nk). \tag{50}$$

Proof: From (48), it follows that

$$\begin{aligned}2\pi\delta(\theta - i(v+nk)) &= 2\pi\delta\left[\frac{1}{i}(i\theta + (v+nk))\right] \\ &= 2\pi|i|\delta(v + i\theta + nk) = 2\pi\delta(z+nk),\end{aligned}$$

hence, it implies that

$$\Gamma_k(z) = 2\pi \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{k}\right)^n}{n!} \delta(\theta - i(v+nk)) = 2\pi \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{k}\right)^n}{n!} \delta(z+nk).$$

□

Corollary 0.3. *(see* [32]*). The k-Gamma function has the subsequent demonstration:*

$$\Gamma_k(z) = 2\pi \sum_{n,r=0}^{\infty} \frac{\left(-\frac{1}{k}\right)^n}{n!} \frac{(nk)^r}{r!} \delta^{(r)}(z). \tag{51}$$

Proof: This is achieved by considering $\delta(z+b) = \sum_{i=0}^{\infty} \delta^{(i)}(z) \frac{b^i}{i!}$, we have

$$\delta(z+nk) = \sum_{r=0}^{\infty} \frac{(nk)^r}{r!} \delta^{(r)}(z),$$

so, from (50), we get

$$\Gamma_k(z) = 2\pi \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{k}\right)^n}{n!} \delta(z+nk) = 2\pi \sum_{n,r=0}^{\infty} \frac{\left(-\frac{1}{k}\right)^n}{n!} \frac{(nk)^r}{r!} \delta^{(r)}(z).$$

□

The following results can be respected obtained by replacing k in the above represents

Corollary 0.4. *(see* [13]*). The classical Gamma function satisfies the following representations*

$$\Gamma(z) = \Gamma_1(z) = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \delta(\theta - i(v+n)). \tag{52}$$

$$\Gamma(z) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-1)^n}{n!} \frac{(-i(v+n))^r}{r!} \delta^{(r)}(\theta). \tag{53}$$

$$\Gamma(z) = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \delta(z+n). \tag{54}$$

$$\Gamma(z) = 2\pi \sum_{n,r=0}^{\infty} \frac{(-1)^n}{n!} \frac{(n)^r}{r!} \delta^{(r)}(z). \tag{55}$$

5|Conclusion

In this paper, we have presented an introductory and systematic overview of the k-analogue of the classical Gamma function and several associated special functions. Particular attention has been devoted to the fundamental definitions, integral representations, analytical properties, and functional identities of the k-Gamma function. By establishing its relationship with the classical Gamma function, we have demonstrated that the parameter k provides a natural framework for generalizing several well-known properties of classical special functions.

Furthermore, the study has examined the principal properties of the k-Beta function, including its integral and product representations, recurrence relations, and connections with the k-Gamma function. The definitions and functional equations of the k-Digamma and k-Polygamma functions have also been discussed. In addition, the lower and upper k-incomplete Gamma functions have been introduced, and their representations in terms of the corresponding classical incomplete Gamma functions have been established.

Another important aspect of this work is the consideration of series and distributional representations of the k-Gamma function. These representations provide alternative analytical perspectives and contribute to a broader understanding of the structural properties of generalized special functions. The obtained relationships also illustrate how classical results can be recovered as particular cases when the parameter k approaches unity.

Overall, this study provides a unified theoretical foundation for further investigations of generalized Gamma-type functions and their extensions. The presented results may serve as a useful starting point for future research in fractional calculus, mathematical analysis, probability theory, and mathematical physics, particularly in developing new inequalities, integral transforms, and applications involving generalized special functions.

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