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Integral Representations of Solutions to an Overdetermined System with Two Internal Singular Lines

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Abstract

In this article, integral representations of solutions are obtained for an overdetermined system of second-order differential equations with two internal singular lines, under a specified relation between the coefficients of the first and second equations. The solution manifold is represented through one arbitrary constant by applying a methodology developed for overdetermined systems with singular and supersingular coefficients. Two cases are considered, depending on which equation is taken as the principal equation. The corresponding solution representations are derived, and several properties of the solutions are established. In addition, two initial value problems are formulated and uniqueness of their solutions is proved under the stated conditions. The results provide integral representations and solvability criteria for a class of overdetermined differential systems with two internal singular lines, contributing to the analysis of differential equations with singular structures.

Keywords: Overdetermined system, Solution manifold, Rectangle, Internal singular lines, Properties of solutions.

1 | Introduction

One of the aĸtyal areas of modern partial differential equation theory is the study of overdetermined systems with regular, singular, and supersingular coefficients. The foundations of the theory of such systems were laid in the works of Jacobi and further developed in studies devoted to linear homogeneous and inhomogeneous first-order systems with one or several unknown functions. An important role in the formation of this research area was played by the development of the Cauchy–Kovalevskaya theory and Frobenius theory of systems in total differentials, within which various classes of analytic overdetermined systems are studied, along with questions concerning the existence and construction of their solution manifolds.

The study of particular classes of overdetermined systems of differential equations with regular, singular, and supersingular coefficients has been addressed in the works of researchers, as well as their respective research

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schools. Initially, the main results in this field were obtained for systems with regular coefficients; subsequently, research was extended to systems involving singular and supersingular coefficients.

Despite the progress achieved and the considerable number of published studies, several issues concerning the behavior of solutions in the vicinity of internal singularities remain insufficiently investigated. In particular, the study of overdetermined systems of differential equations with two internal singular lines is of particular relevance, since the presence of multiple internal singular lines significantly complicates the structure of the system and the formulation of problems related to the construction of its solution manifold. These limitations highlight the need for further investigation of such systems and the development of appropriate methods for their analysis.

Let us denote:

$$\Gamma_1^0 = \{y = x, 0 \le x \le a\}, \Gamma_2^0 = \{y = -x, -a \le x \le 0\}.$$

In the domain D

$$\begin{cases} \dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{a_1(x,y)}{(x^2-y^2)^m}\dfrac{\partial u}{\partial x} + \dfrac{b_1(x,y)}{(x^2-y^2)^n}\dfrac{\partial u}{\partial y} + \\ \quad + \dfrac{c_1(x,y)}{(x^2-y^2)^{m+n}}u = \dfrac{f_1(x,y)}{(x^2-y^2)^{m+n}}, \\ \quad \dfrac{\partial u}{\partial y} + \dfrac{b_2(x,y)}{(x^2-y^2)^p}u = \dfrac{f_2(x,y)}{(x^2-y^2)^p}, \end{cases} \tag{1}$$

where the functions are prescribed in the domain, the parameters are natural numbers, and the function denotes the unknown function. $a_1(x,y), b_i(x,y),\ c_1(x,y), f_i(x,y), i = \overline{1,2}\ - D, m, n, p - u(x,y)$.

The study of differential equations and overdetermined systems with regular, singular, and supersingular coefficients has been addressed in [1–12].

Using the methodology developed in [5], [6] for *System (1)*, a representation of the solution manifold in terms of one arbitrary constant is obtained.

Case 1. Suppose that the first equation of *System (1)* is taken as the principal equation. Then its solution can be written in the form. In this case, the first equation of *System (1)* can be represented as

$$\left(\frac{\partial}{\partial y} + \frac{a_1(x,y)}{(x^2-y^2)^m}\right)\left(\frac{\partial}{\partial x} + \frac{b_1(x,y)}{(x^2-y^2)^n}\right)u = \frac{f_1(x,y) + c_2(x,y)u(x,y)}{(x^2-y^2)^{m+n}}, \tag{2}$$

where

$$c_2(x,y) = -c_1(x,y) + (x^2-y^2)^{m+n}\frac{\partial}{\partial y}\left(\frac{b_1(x,y)}{(x^2-y^2)^n}\right) + a_1(x,y)b_1(x,y). \tag{3}$$

Introducing a new unknown function by the formula $\vartheta_1(x,y)$:

$$\frac{\partial u}{\partial x} + \frac{b_1(x,y)}{(x^2-y^2)^n}u = \vartheta_1(x,y), \tag{4}$$

We reduce the problem to solving the following first-order differential equation:

$$\frac{\partial \vartheta_1}{\partial y} + \frac{a_1(x,y)}{(x^2-y^2)^m}\vartheta_1 = \frac{f_1(x,y) + c_2(x,y)u(x,y)}{(x^2-y^2)^{m+n}}. \tag{5}$$

The solution of *Eq. (5)*, according to [5], can be written in the form:

$$\vartheta_1(x,y)=\exp\left[-W^m_{a_1}(x,y)-a_1(x,x)J^{(1)}_{m-1}(x,y)\right]$$

$$\times\left(\varphi_1(x)+\int_0^y\frac{f_1(x,s)+c_2(x,s)u(x,s)}{(x^2-s^2)^{m+n}}\exp\left[W^m_{a_1}(x,s)+a_1(x,x)J^{(1)}_{m-1}(x,s)\right]ds\right), \quad (6)$$

where

$$W^m_{a_1}(x,y)=\int_0^y\frac{a_1(x,s)-a_1(x,x)}{(x^2-s^2)^m}ds,$$

$$J^{(1)}_{m-1}(x,y)=\frac{y}{2(m-1)x^2(x^2-y^2)^{m-1}}+\frac{2m-3}{2(m-1)x^2}\int_0^y\frac{ds}{(x^2-s^2)^{m-1}}.$$

Now, solving *Eq. (4)*, we express the function in terms of $u(x,y)$:

$$u(x,y)=\exp\left[-W^n_{b_1}(x,y)+b_1(y,y)J^{(2)}_{n-1}(x,y)\right]$$

$$\times\left(\psi_1(y)+\int_0^x\vartheta_1(t,y)\exp\left[W^n_{b_1}(t,y)-b_1(y,y)J^{(2)}_{n-1}(t,y)\right]dt\right), \quad (7)$$

where

$$W^n_{b_1}(x,y)=\int_0^x\frac{b_1(t,y)-a_1(y,y)}{(t^2-y^2)^n}dt,$$

$$J^{(2)}_{n-1}(x,y)=\frac{x}{2(n-1)y^2(x^2-y^2)^{n-1}}+\frac{2n-3}{2(n-1)y^2}\int_0^x\frac{dt}{(t^2-y^2)^{n-1}}.$$

Substituting its value from *Eq. (6)* into *Eq. (7)*, we obtain $\vartheta_1(x,y)$:

$$u(x,y)=\exp\left[-W^n_{b_1}(x,y)+b_1(y,y)J^{(2)}_{n-1}(x,y)\right]$$

$$\times\left\{\psi_1(y)+\int_0^x\exp\left[W^n_{b_1}(t,y)-b_1(y,y)J^{(2)}_{n-1}(t,y)-W^m_{a_1}(t,y)-a_1(t,t)J^{(1)}_{m-1}(t,y)\right]\right.$$

$$\left.\times\left(\varphi_1(t)+\int_0^y\frac{f_1(t,s)+c_2(t,s)u(t,s)}{(t^2-s^2)^{m+n}}\exp\left[W^m_{a_1}(t,s)+a_1(t,t)J^{(1)}_{m-1}(t,s)\right]ds\right)dt\right\}\equiv \quad (8)$$

$$K_1\left(\psi_1(y),\varphi_1(x),f_1(x,y)\right).$$

The second equation of *System (1)* can be written in the form:

$$\frac{\partial}{\partial y}\left\{\exp\left[W^1_{b_2}(x,y)+\frac{b_2(x,x)}{2x}\ln\left|\frac{x+y}{x-y}\right|\right]u(x,y)\right\} \quad (9)$$

$$= \frac{f_2(x,y)}{x^2-y^2}\exp\left[W^1_{b_2}(x,y)+\frac{b_2(x,x)}{2x}\ln\left|\frac{x+y}{x-y}\right|\right],$$

where

$$W^1_{b_2}(x,y)=\int_0^y \frac{b_2(x,s)-b_2(x,x)}{x^2-s^2}\,ds.$$

Substituting its value from *Eq. (9)* into *Equality (8)*, we obtain $u(x,y)$:

$$\begin{aligned}&\frac{\partial}{\partial y}\left\{\exp\left[W^1_{b_2}(x,y)+\frac{b_2(x,x)}{2x}\ln\left|\frac{x+y}{x-y}\right|-W^n_{b_1}(x,y)+b_1(y,y)J^{(2)}_{n-1}(x,y)\right]\right.\\&\times\left\{\psi_1(y)+\int_0^x \exp\left[W^n_{b_1}(t,y)-b_1(y,y)J^{(2)}_{n-1}(t,y)-W^m_{a_1}(t,y)-a_1(t,t)J^{(1)}_{m-1}(t,y)\right]\right.\\&\left.\left.\times\left(\varphi_1(t)+\int_0^y \frac{f_1(t,s)+c_2(t,s)u(t,s)}{(t^2-s^2)^{m+n}}\exp\left[W^m_{a_1}(t,s)+a_1(t,t)J^{(1)}_{m-1}(t,s)\right]ds\right)dt\right\}\\&=\frac{f_2(x,y)}{x^2-y^2}\exp\left[W^1_{b_2}(x,y)+\frac{b_2(x,x)}{2x}\ln\left|\frac{x+y}{x-y}\right|\right].\end{aligned} \tag{10}$$

Before differentiating *Equality (10)*, we first require that the following condition be satisfied:

$$\frac{\partial}{\partial y}\left(\frac{b_1(x,y)}{(x^2-y^2)^n}\right)=\frac{\partial}{\partial x}\left(\frac{b_2(x,y)}{x^2-y^2}\right)\ \text{in}\ D, \tag{11}$$

Differentiating *Equality (10)*, we obtain:

$$\begin{aligned}&-\frac{b_2(0,y)}{y^2}\exp\left[W^1_{b_2}(x,y)+\frac{b_2(x,x)}{2x}\ln\left|\frac{x+y}{x-y}\right|-W^n_{b_1}(x,y)+b_1(y,y)J^{(2)}_{n-1}(x,y)\right]\\&\times\left\{\psi_1(y)+\int_0^x \exp\left[W^n_{b_1}(t,y)-b_1(y,y)J^{(2)}_{n-1}(t,y)-W^m_{a_1}(t,y)-a_1(t,t)J^{(1)}_{m-1}(t,y)\right]\right.\\&\left.\times\left(\varphi_1(t)+\int_0^y \frac{f_1(t,s)+c_2(t,s)u(t,s)}{(t^2-s^2)^{m+n}}\exp\left[W^m_{a_1}(t,s)+a_1(t,t)J^{(1)}_{m-1}(t,s)\right]ds\right)dt\right\}\\&=\frac{f_2(x,y)}{x^2-y^2}\exp\left[W^1_{b_2}(x,y)+\frac{b_2(x,x)}{2x}\ln\left|\frac{x+y}{x-y}\right|\right].\end{aligned} \tag{12}$$

Dividing both sides of *Equality (12)* by

$$\exp\left[W^1_{b_2}(x,y)+\frac{b_2(x,x)}{2x}\ln\left|\frac{x+y}{x-y}\right|-W^n_{b_1}(x,y)+b_1(y,y)J^{(2)}_{n-1}(x,y)\right],$$

and transferring some terms to the right-hand side, we obtain:

$$\psi_1'(y)-\frac{b_2(0,y)}{y^2}\psi_1(y)=\frac{f_2(x,y)}{x^2-y^2}\exp\left[W^n_{b_1}(x,y)-b_1(y,y)J^{(2)}_{n-1}(x,y)\right]$$

$$+\frac{b_2(0,y)}{y^2}\int_0^x \exp\left[W^n_{b_1}(t,y)-b_1(y,y)J^{(2)}_{n-1}(t,y)-W^m_{a_1}(t,y)-a_1(t,t)J^{(1)}_{m-1}(t,y)\right]$$

$$\times\left(\varphi_1(t)+\int_0^y\frac{f_1(t,s)+c_2(t,s)u(t,s)}{(t^2-s^2)^{m+n}}\exp\left[W^m_{a_1}(t,s)+a_1(t,t)J^{(1)}_{m-1}(t,s)\right]ds\right)dt- \qquad (13)$$

$$-\frac{\partial}{\partial y}\int_0^x \exp\left[W^n_{b_1}(t,y)-b_1(y,y)J^{(2)}_{n-1}(t,y)-W^m_{a_1}(t,y)-a_1(t,t)J^{(1)}_{m-1}(t,y)\right]$$

$$\times\left(\varphi_1(t)+\int_0^y\frac{f_1(t,s)+c_2(t,s)u(t,s)}{(t^2-s^2)^{m+n}}\exp\left[W^m_{a_1}(t,s)+a_1(t,t)J^{(1)}_{m-1}(t,s)\right]ds\right)dt.$$

From the condition that the left-hand side of *Eq. (13)* is independent of the variable x, we obtain:

$$\frac{\partial}{\partial y}\left\{\frac{f_2(x,y)}{x^2-y^2}\exp\left[W^n_{b_1}(x,y)-b_1(y,y)J^{(2)}_{n-1}(x,y)\right]\right\}$$

$$-\frac{\partial}{\partial x}\left\{\exp\left[W^n_{b_1}(x,y)-b_1(y,y)J^{(2)}_{n-1}(x,y)-W^m_{a_1}(x,y)-a_1(x,x)J^{(1)}_{m-1}(x,y)\right]\right.$$

$$\left.\times\left(\varphi_1(x)+\int_0^y\frac{f_1(x,s)+c_2(x,s)u(x,s)}{(x^2-s^2)^{m+n}}\exp\left[W^m_{a_1}(x,s)+a_1(x,x)J^{(1)}_{m-1}(x,s)\right]ds\right)\right\} \qquad (14)$$

$$=-\frac{b_2(0,y)}{y^2}\exp\left[W^n_{b_1}(x,y)-b_1(y,y)J^{(2)}_{n-1}(x,y)-W^m_{a_1}(x,y)-a_1(x,x)J^{(1)}_{m-1}(x,y)\right]$$

$$\times\left(\varphi_1(x)+\int_0^y\frac{f_1(x,s)+c_2(x,s)u(x,s)}{(x^2-s^2)^{m+n}}\exp\left[W^m_{a_1}(x,s)+a_1(x,x)J^{(1)}_{m-1}(x,s)\right]ds\right).$$

Transforming the last integral in *Equality (13)* according to *Eq. (14)*, we obtain:

$$\frac{\partial}{\partial y}\int_0^x \exp\left[W^n_{b_1}(t,y)-b_1(y,y)J^{(2)}_{n-1}(t,y)-W^m_{a_1}(t,y)-a_1(t,t)J^{(1)}_{m-1}(t,y)\right]$$

$$\times\left(\varphi_1(t)+\int_0^y\frac{f_1(t,s)+c_2(t,s)u(t,s)}{(t^2-s^2)^{m+n}}exp\left[W^m_{a_1}(t,s)+a_1(t,t)J^{(1)}_{m-1}(t,s)\right]ds\right)dt$$

$$=\int_0^x\frac{\partial}{\partial t}\left\{\frac{f_2(t,y)}{t^2-y^2}\exp\left[W^n_{b_1}(t,y)-b_1(y,y)J^{(2)}_{n-1}(t,y)\right]\right\}$$

$$+\frac{b_2(0,y)}{y^2}\int_0^x \exp\left[W_{b_1}^n(t,y)-b_1(y,y)J_{n-1}^{(2)}(t,y)-W_{a_1}^m(t,y)-a_1(t,t)J_{m-1}^{(1)}(t,y)\right]$$

$$\times\left(\varphi_1(t)+\int_0^y \frac{f_1(t,s)+c_2(t,s)u(t,s)}{(t^2-s^2)^{m+n}}\exp\left[W_{a_1}^m(t,s)+a_1(t,t)J_{m-1}^{(1)}(t,s)\right]ds\right)dt$$

$$=\frac{f_2(x,y)}{x^2-y^2}\exp\left[W_{b_1}^n(x,y)-b_1(y,y)J_{n-1}^{(2)}(x,y)\right]+\frac{f_2(0,y)}{y^2}$$

$$+\frac{b_2(0,y)}{y^2}\int_0^x \exp\left[W_{b_1}^n(t,y)-b_1(y,y)J_{n-1}^{(2)}(t,y)-W_{a_1}^m(t,y)-a_1(t,t)J_{m-1}^{(1)}(t,y)\right]$$

$$\times\left(\varphi_1(t)+\int_0^y \frac{f_1(t,s)+c_2(t,s)u(t,s)}{(t^2-s^2)^{m+n}}\exp\left[W_{a_1}^m(t,s)+a_1(t,t)J_{m-1}^{(1)}(t,s)\right]ds\right)dt.$$

Substituting the value of the transformed integral into *Eq. (12)*, we obtain:

$$\psi_1'(y)-\frac{b_2(0,y)}{y^2}\psi_1(y)=-\frac{f_2(0,y)}{y^2}. \tag{15}$$

The solution of *Eq. (15)* can be written in the form

$$\psi_1(y)=\exp\left[W_{b_2}^2(0,y)-b_2(0,0)W_1(y)\right]\times$$

$$\left(c_1-\int_0^y \frac{f_2(0,s)}{s^2}\exp\left[-W_{b_2}^2(0,s)+b_2(0,0)W_1(s)\right]ds\right)\equiv N_1\left(c_1,f_2(0,y)\right), \tag{16}$$

where

$$W_{b_2}^2(0,y)=\int_0^y \frac{b_2(0,s)-b_2(0,0)}{s^2}ds,\quad W_1(y)=\frac{1}{y},$$

c_1 –is an arbitrary constant.

We now simplify *Equality (14)*. For this purpose, differentiating it and making some simplifications, we obtain:

$$[(x^2-y^2)^{m+n}b_2(x,y)-(x^2-y^2)^{n+1}a_1(x,y)]$$

$$\times\exp\left[-W_{a_1}^m(x,y)-a_1(x,x)J_{m-1}^{(1)}(x,y)\right]$$

$$\times\left(\varphi_1(x)+\int_0^y \frac{f_1(x,s)+c_2(x,s)u(x,s)}{(x^2-s^2)^{m+n}}\exp\left[W_{a_1}^m(x,s)+a_1(x,x)J_{m-1}^{(1)}(x,s)\right]ds\right) \tag{17}$$

$$+(x^2-y^2)f_1(x,y)+(x^2-y^2)c_2(x,y)u(x,y)$$

$$= (x^2-y^2)^{m+n+1}\frac{\partial}{\partial x}\left(\frac{f_2(x,y)}{x^2-y^2}\right) + (x^2-y^2)^m b_1(x,y)f_2(x,y) \text{ в } D.$$

Passing to the limit in *Equality (17)*, we obtain:$y \to 0$.

$$\varphi_1(x) = \frac{1}{x^{2(m+n)}b_2(x,0) - x^{2(n+1)}a_1(x,0)}\left[x^{2(m+n-1)}\frac{\partial}{\partial x}\left(\frac{f_2(x,y)}{x^2-y^2}\right)\Big|_{y=0} + x^{2m}b_1(x,0)f_2(x,0) - x^2 f_1(x,0)\right], \quad x^{2(m+n)}b_2(x,0) - x^{2(n+1)}a_1(x,0). \tag{18}$$

Thus, the following theorem is proved.

Theorem 1. Suppose that, in the system of differential *Eqs. (1)*, $m \geq 2, n \geq 2, p = 1$ and that the following conditions are satisfied by the coefficients and the right-hand sides:

I. $b_1(x,y) \in C_y^1(\overline{D})$, $b_2(x,y) \in C_x^1(\overline{D}), f_2(x,y) \in C_x^1(\overline{D}), a_1(x,y)$.

$c_1(x,y)$, $f_1(x,y) \in C(\overline{D})$.

II. $c_1(x,y) = (x^2-y^2)^{m+n}\frac{\partial}{\partial y}\left(\frac{b_1(x,y)}{(x^2-y^2)^n}\right) + a_1(x,y)b_1(x,y)$.

III. $b_1(y,y) > 0, a_1(x,x) > 0$, $b_2(0,0) < 0$.

IV. $b_1(x,y) - b_1(y,y) = o\big((x-y)^{\beta_1}\big)$, $\beta_1 > n-1$ in a neighborhood of Γ_1^0.

$b_1(x,y) - b_1(y,y) = o\big((x+y)^{\beta_2}\big)$, $\beta_2 > n-1$ in a neighborhood of Γ_2^0.

$a_1(x,y) - a_1(x,x) = o((x-y)^{\alpha_1})$, $\alpha_1 > m-1$ in a neighborhood of Γ_1^0.

$a_1(x,y) - a_1(x,x) = o((x+y)^{\alpha_2})$, $\alpha_2 > m-1$ in a neighborhood of Γ_2^0.

$|b_2(0,y) - b_2(0,0)| \leq H_1 y^{\mu_1}$, $H_1 = \text{const}$, $\mu_1 > 1$.

V. a) $\frac{\partial}{\partial y}\left(\frac{b_1(x,y)}{(x^2-y^2)^n}\right) = \frac{\partial}{\partial x}\left(\frac{b_2(x,y)}{x^2-y^2}\right)$ in D.

b) $[(x^2-y^2)^{m+n}b_2(x,y) - (x^2-y^2)^{n+1}a_1(x,y)]$

$\times \exp\left[-W_{a_1}^m(x,y) - a_1(x,x)J_{m-1}^{(1)}(x,y)\right]\left(\varphi_1(x) + \int_0^y \frac{f_1(x,s)}{(x^2-s^2)^{m+n}}\right.$

$\left.\times \exp\left[W_{a_1}^m(x,s) + a_1(x,x)J_{m-1}^{(1)}(x,s)\right]ds\right) + (x^2-y^2)f_1(x,y) + (x^2-y^2)$

$= (x^2-y^2)^{m+n+1}\frac{\partial}{\partial x}\left(\frac{f_2(x,y)}{x^2-y^2}\right) + (x^2-y^2)^m b_1(x,y)f_2(x,y)$ in D.

VI. $f_1(x,y) = o\big(\exp\left[-a_1(x,x)J_{m-1}^{(1)}(x,y)\right](x-y)^{\lambda_1}\big)$, $\lambda_1 > m+n-1$.

in a neighborhood of Γ_1^0,

$$f_1(x,y) = o\left(\exp\left[-a_1(x,x)J_{m-1}^{(1)}(x,y)\right](x+y)^{\lambda_2}\right), \qquad \lambda_2 > m+n-1,$$

in a neighborhood of Γ_2^0,

$$f_2(0,y) = o(y^{\mu_2}), \quad \mu_2 > 1.$$

Then, under the above requirements, every solution of the considered *System (1)* belonging to the class $C^2(D \setminus (\Gamma_0^1 \cup \Gamma_0^2))$ can be found from *Formulas (8), (16),* and *(18)*.

Let us formulate some properties of the obtained solution:

I. If we assume that, then we have $x \to 0$,

$u(0,y) = \psi_1(y).$

II. For $x \to 0$ and $y \to 0$, we obtain the estimate.

$$\lim_{y\to 0}\left\{\lim_{x\to 0} u(x,y)\right\} = O(\exp[-b_2(0,0)W_1(y)]).$$

I. $\lim_{y\to 0}\left\{\exp[b_2(0,0)W_1(y)]\lim_{y\to 0} u(x,y)\right\} = c_1.$

II. If, then $y \to x$.

III. $u(x,y) = O\left(\exp\left[b_1(y,y)J_{n-1}^{(2)}(x,y)\right]\right)$ in a neighborhood of Γ_1^0.

IV. If, then $y \to -x$.

$u(x,y) = O\left(\exp\left[b_1(y,y)J_{n-1}^{(2)}(x,y)\right]\right)$ in a neighborhood of Γ_2^0.

Case 2. Consider the case when the second equation of *System (1)* is taken as the principal equation. Using the above scheme, we obtain the following.

Theorem 2. Let $m \geq 2, n \geq 2, p = 1$ in the system of *Eq. (1)*. In addition, suppose that the coefficients and right-hand sides satisfy the following conditions:

I. $b_1(x,y) \in C_y^1(\bar{D}),\ b_2(x,y) \in C_x^1(\bar{D}), f_2(x,y) \in C_x^1(\bar{D}), a_1(x,y),$

$c_1(x,y), f_1(x,y) \in C(\bar{D}).$

II. $c_1(x,y) = (x^2-y^2)^{m+n}\frac{\partial}{\partial y}\left(\frac{b_1(x,y)}{(x^2-y^2)^n}\right) + a_1(x,y)b_1(x,y).$

III. $\frac{b_2(x,x)}{2x} < 0$ in a neighborhood of Γ_1^0.

$\frac{b_2(x,x)}{2x} > 0$ in a neighborhood of Γ_2^0,

$B_1(0,0) > 0.$

IV. $|b_2(x,y) - b_2(x,x)| \leq H_1|x-y|^{\mu_1},\ H_1 = \text{const},\ \mu_1 > 1.$

– *in a neighborhood of Γ_1^0,*

– *|b_2 (x,y)-b_2 (x,x)|≤H_2 |x+y|^(μ_2), H_2=const, μ_2>1*

– *in a neighborhood of Γ_2^0,*

– *|B_1 (x,0) -B_1 (0,0) |≤H_3 x^(μ_3), H_3=const, μ_3>2(m+n)+1;*

V. $\frac{\partial}{\partial x}\left(\frac{b_2(x,y)}{x^2-y^2}\right) = \frac{\partial}{\partial y}\left(\frac{b_1(x,y)}{(x^2-y^2)^n}\right)$ in D.

$$(x^2-y^2)^{m+n+1}\left((x^2-y^2)^{-1}b_2(x,y) - (x^2-y^2)^{-m}a_1(x,y)\right)$$

$$\times \exp\left[-W_{a_1}^m(x,y) - a_1(x,x)J_{m-1}^{(1)}(x,y)\right]\left(\varphi_1(x) + \int_0^y \frac{f_1(x,s) + c_2(x,s)\Omega_2\left(\varphi_2(x), f_2(x,s)\right)}{(x^2-s^2)^{m+n}}\right.$$

$$\left.\times \exp\left[W_{a_1}^m(x,s) + a_1(x,x)J_{m-1}^{(1)}(x,s)\right]ds\right) + (x^2-y^2)f_1(x,y)$$

$$+(x^2-y^2)c_2(x,y)\Omega_2\left(\varphi_2(x), f_2(x,s)\right) =$$

$$= (x^2-y^2)^m b_1(x,y)f_2(x,y) + (x^2-y^2)^{m+n+1}\frac{\partial}{\partial x}\left(\frac{f_2(x,y)}{x^2-y^2}\right) \text{ in D.}$$

VI. $f_2(x,y) = o\left((x-y)^{\theta_1}\right),\ \theta_1 > \frac{b_2(x,x)}{2x}$ in a neighborhood of Γ_1^0.

$f_2(x,y) = o\big((x+y)^{\theta_2}\big),\ \theta_2 > -\frac{b_2(x,x)}{2x}$ in a neighborhood of Γ_2^0.

$F_1(x) = o(x^{\gamma_1}),\ \gamma_1 > 2(m+n)+1.$

Then, under the above assumptions, any solution of the *System (1)* under consideration belonging to the class $C^2(D \setminus (\Gamma_0^1 U \Gamma_0^2))$ can be found from the formula:

$$u(x,y) = \exp\big[-W_{b_2}^1(x,y)\big]\left|\frac{x+y}{x-y}\right|^{-\frac{b_2(x,x)}{2x}}$$

$$\times\left(\varphi_2(x) + \int_0^y \frac{f_2(x,s)}{x^2-s^2}\exp\big[W_{b_2}^1(x,s)\big]\left|\frac{x+s}{x-s}\right|^{\frac{b_2(x,x)}{2x}} ds\right) \quad (19)$$

$$\equiv K_2\big(\varphi_2(x), f_2(x,y)\big),$$

$$\varphi_2(x) = \exp\left[-W_{B_1}^{2(m+n+1)}(x,0) + B_1(0,0)W_{2(m+n)+1}(x)\right]$$

$$\times\left(c_2 + \int_0^x \frac{F_1(t)}{t^{2(m+n+1)}}\exp\left[W_{B_1}^{2(m+n+1)}(t,0) - B_1(0,0)W_{2(m+n)+1}(t)\right] dt\right) \quad (20)$$

$$\equiv N_2\big(c_2, F_1(x)\big),$$

where

$$W_{b_2}^1(x,y) = \int_0^y \frac{b_2(x,s)-b_2(x,x)}{x^2-s^2} ds,\ W_{B_1}^{2(m+n+1)}(x,0) = \int_0^x \frac{B_1(t,0)-B_1(0,0)}{t^{2(m+n+1)}} dt,$$

$$W_{2(m+n)+1}(x) = \frac{1}{(2(m+n)+1)x^{2(m+n)+1}},$$

$$F_1(x) = \frac{1}{x^{-2}b_2(x,0) - x^{-2m}a_1(x,0)}$$

$$\times\left[x^{2m}b_1(x,0)f_2(x,0) + x^{2(m+n+1)}\frac{\partial}{\partial x}\left(\frac{f_2(x,y)}{x^2-y^2}\right)\Big|_{y=0} - x^2 f_1(x,0)\right]$$

$$\big(x^{-2m}a_1(x,0) \neq x^{-2}b_2(x,0)\big),$$
$$\big(x^{-2m}a_1(x,0) \neq x^{-2}b_2(x,0)\big),$$

c_2 –is an arbitrary constant.

The obtained solution has a number of properties. Some of them are listed below.

I. If we assume that, then we have $y \to 0$,

$$u(x,0) = \varphi_2(x).$$

II. For and, we have the estimate $y \to 0 x \to 0$.

$$\lim_{x\to 0}\left\{\lim_{y\to 0} u(x,y)\right\} = O\big(\exp\big[B_1(0,0)W_{2(m+n)+1}(x)\big]\big).$$

I. $\lim\limits_{x\to 0}\left\{\exp\left[-B_1(0,0)W_{2(m+n)+1}(x)\right]\lim\limits_{y\to 0}u(x,y)\right\}=c_2.$

II. If and, then $x\to 0 y\neq 0.$

$$u(x,y)=O\left(\exp\left[B_1(0,0)W_{2(m+n)+1}(x)\right]\right).$$

III. If, then $y\to x.\, u(x,y)=O\left(\left|\frac{x+y}{x-y}\right|^{-\frac{b_2(x,x)}{2x}}\right)$ in a neighborhood of Γ_1^0.

IV. If, then $y\to -x.\, u(x,y)=O\left(\left|\frac{x+y}{x-y}\right|^{-\frac{b_2(x,x)}{2x}}\right)$ in a neighborhood of Γ_2^0.

Problem K_1. Find a solution of *System (1)* belonging to the class $C^2\left(D\setminus(\Gamma_0^1\cup\Gamma_0^2)\right)$ in the domain D, for which the condition is satisfied: $m\geq 2, n\geq 2, p=1.$

$$\lim_{y\to 0}\left\{\exp[b_2(0,0)W_1(y)]\lim_{y\to 0}u(x,y)\right\}=p_1,$$

where p_1 −a known constant is prescribed.

Problem K_2. Under the conditions, find a solution of *System (1)* belonging to the class $C^2\left(D\setminus(\Gamma_0^1\cup\Gamma_0^2)\right)$ in the domain D and satisfying the requirement $m\geq 2, n\geq 2, p=1.$

$$\lim_{x\to 0}\left\{\exp\left[-B_1(0,0)W_{2(m+n)+1}(x)\right]\lim_{y\to 0}u(x,y)\right\}=p_2,$$

where p_2 −a given known constant is assumed.

Theorem 3. Suppose that the coefficients and right-hand sides of *System (1)* satisfy the conditions of *Theorem 1*. Then *Problem* K_1 has a unique solution determined by *Relations (8), (16),* and *(18)*, with $c_1=p_1$.

Theorem 4. Under the conditions of *Theorem 2* for *System (1)*, *Problem* K_2 has a unique solution determined by *Relations (19)* and *(20)*, in which $c_2=p_2$.

2 | Conclusion

In this paper, we investigated an overdetermined system of second-order differential equations with two internal singular lines, under appropriate assumptions on the coefficients and the right-hand sides of the system. By employing an integral representation method developed for differential equations with singular and supersingular coefficients, explicit representations of the solution manifold were established in two different cases, depending on the choice of the principal equation.

The main results demonstrate that, under suitable regularity, compatibility, and asymptotic conditions, the solutions of the considered system can be expressed in terms of integral formulas involving one arbitrary constant. In particular, *Theorems 1* and *2* provide explicit representations of the solutions when the first and the second equations are respectively selected as the principal equations. The compatibility conditions imposed on the coefficients and the right-hand sides ensure the consistency of the two equations and play a fundamental role in the construction of the solution manifold.

Furthermore, several important properties of the obtained solutions were established, including their asymptotic behavior near the internal singular lines (y=x) and (y=-x), as well as their limiting behavior near the origin. These results clarify how the singular coefficients influence the growth and structure of solutions in the neighborhoods of the singular lines.

As an application of the derived integral representations, two initial value *problems, K_1* and *K_2*, were formulated and investigated. *Theorems 3* and *4* establish the existence and uniqueness of solutions to these problems under the corresponding assumptions. In both cases, prescribing the appropriate limiting value

determines the arbitrary constant appearing in the general solution representation, thereby yielding a unique solution.

The results obtained extend the study of overdetermined differential systems with singular structures to the case of two internal singular lines and provide a constructive framework for investigating their solvability and solution properties. They may also serve as a basis for further research on systems with several internal singular lines, higher-order singularities, and more general classes of singular and supersingular coefficients.

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All aspects of the research and manuscript preparation were carried out by the author. The author has read and approved the final version of the manuscript.

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